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ABSTRACT

This is volume one of a two-volume manual for teachers using SMSG text materials for students in grades 7 and 8 whose mathematical talents are underdeveloped. The overall purpose for each of the chapters is described and the mathematical development detailed. Background information for key concepts, answers for all exercises in each chapter, and suggested test items are provided. Chapter topics include: number symbols, whole numbers, non-metric geometry, factoring and primes, rational numbers and fractions, rational numbers and the number line, decimals, and ratio and percent. (MN)

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TEACHER'S COMMENTARY UNIT NO. 39

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INTRODUCTION TO SECONDARY SCHOOL MATHEMATICS VOLUME 1

SE 024 959

SCHOOL MATHEMATICS STUDY GROUP

YALE UNIVERSITY PRESS

Introduction to Secondary School Mathematics, Volume 1

Teacher's Commentary

REVISED EDITION

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PREFACE TO TEACHERS

This text has been written for the students in Grade 7 whose mathematical talent is underdeveloped. The subject matter presented is essentially that which appears in the School Mathematics Study Group text: "Mathematics for Junior High School", Volume 1. This is part of the body of mathematics which members of the Study Group believe is important for all educated citizens. It is also the mathematics which is important for the pre-college student as he prepares for advanced work in the field of mathematics and related subjects.

Within the group for which this material is intended there may be a large number of college capable students whose mathematical talent as yet has not been discovered. There may be others who heretofore have been insufficiently challenged. This text is not offered as appropriate content for the very slow non-college-bound student.

A number of guides were followed in preparing these materials, among which are the following:

- to adjust the reading level downward;
- to shorten the chapters and to provide variation from chapter to chapter in terms of content;
- to shorten sections within each chapter;
- to introduce new concepts through the use of concrete examples;
- to provide numerous illustrative examples;
- to include simple drill material in many of the problem sets;
- to provide chapter summaries, chapter reviews, and cumulative sets of problems.

The mathematics which appears in this text is not of the type normally called "business" or "vocational" mathematics, nor is it intended to serve as the content for a terminal course. Rather, as the title clearly states, this is an introduction to secondary school mathematics which will provide the student with many of the basic concepts necessary for further study.

It is the hope of the panel that this material will serve to awaken the interest of a large group of students who have ability in mathematics which has not yet been recognized. It is hoped also that an understanding of fundamental concepts can be built for those whose progress in mathematics has been blocked or hampered through rote learning or through an inappropriate curriculum.

The teacher should keep in mind that this is an experimental text which is being used to test the hypothesis that material of this type can be taught to young people of the ability level previously described. Consequently the development should follow the text closely in terms of content as well as methodology in order that a fair evaluation of the material may be made.

The material in this text is presented in a manner different from the usual text at this level, and as previously indicated, is written for a particular type of student. For these reasons some general suggestions for its use are offered below.

Reading. This text is written with the expectation that it can and will be read by the student. Many students are not accustomed to reading a mathematics book so it will be necessary to assist them in learning how to make the most effective use of this text.

Experience has shown that it is most productive for the teacher to read with the class during the early part of the course. The teacher may read aloud while the students read silently. Later the teacher may start the reading with the class and then encourage them to continue the reading alone. This is not recommended as a method of teaching reading as such, but rather as a method of helping the student discover that he can read a mathematics book.

There will be times when the student will need to reread the same passages several times. The teacher should suggest this, and make time available for it.

The students of the ability level for which this text was written may not be able to read long passages with understanding. Some students may be able to read only a sentence or two at a time in the beginning. Consequently, it is important that the teacher stop often for class discussion.

It may prove helpful if students are asked to state, in their own words, the ideas which they have read; but the teacher should remember that some pupils may understand even though they cannot verbalize.

It must be observed that the teacher's objective is to convey to the students the ideas contained in the material. He cannot permit reading retardation to inhibit or to undermine student interest in the content. The mathematics teacher cannot overcome serious pupil retardation in reading, but he can contribute to reading skill by pointing out to the student the need for rereading and giving careful consideration to the material. The use of a pencil and paper to draw diagrams and illustrate ideas should be encouraged.

Precision of Language. Ideally, pupils should be encouraged to express themselves accurately. Some pupils, however, have limited vocabulary resources. It is wise to encourage them to express themselves in their own words, meager as their contributions may be. The pupil's inadequate expression may then be refined by the teacher so that it is mathematically precise. The teacher must also recognize that extreme insistence upon precise formulation may interfere with thought patterns and act as a barrier to free expression.

Discussion Questions. Periodically the text provides discussion questions which are useful in helping to strengthen or emphasize basic concepts and understandings. These are especially useful in developing ideas in sections of the text where straight reading may be difficult. Therefore, where class discussion exercises are provided, they should be treated orally within the class period and not omitted.

Discovery Approach. A student usually gains a better understanding of a concept if he "discovers" the concept himself. The teacher must set the stage for the discovery approach. No textbook can do this because the text must give the student correct information to which he can refer and by which he can check his own ideas. Therefore, the approach will not be effective unless the pupil is encouraged to work through the development before he reads in the text the idea he was to "discover".

Students with limited ability should be given the opportunity to "discover" very simple ideas. For instance, in Section 2-2, the student could be given 23 objects, or a paper with 23 ungrouped marks, and be asked to collect these in groups of 10. He could repeat this with other collections of objects. Thus he will "discover" that the objects are to be arranged in groups of 10 in order to write the correct numeral in the base 10.

It is important for these students to have many experiences with an idea in order to develop meaning. In all cases the teacher will need to clarify the idea which the student has discovered and assist him in finding "his" idea in the text in correct matnematical language.

Exercises. The text has an ample supply of exercises. They are graded in most sections so that the most difficult are at the end. Some of the exercise lists, however, are developmental in nature and need to be treated sequentially. The teacher should be very cautious about making any omissions in such lists. In other lists the teacher may find it desirable to omit selected items.

Assignments. Assignments for this group should be quite definite and should normally concern only material which has been discussed in class so that the student may enjoy some measure of success in the preparation of it. Exercises which demand deeper vision, a higher degree of abstraction, or a preview into new material should be called "extra credit", or given some such notation, so that the student with below-average ability may omit this part of the assignment without any feeling of failure or frustration.

Testing. Students of the ability for which this text is written need to have short tests at frequent intervals--possibly one a week. These tests, like the assignments, should be flexible. The major portion of the test should cover material actually discussed in class with a few exercises for the more capable students included at the end. If the slower learning students are not given some test questions which they can answer correctly, they may lose interest in the course and the opportunity to improve their mathematical background will be lost. They must be permitted to enjoy some measure of success.

Since the intent of this book is to emphasize grasp of ideas rather than memorization, the testing program should be geared accordingly. The teacher should be generous in accepting expressions of ideas in the students' own words.

Extent of Course. The number of chapters studied will depend upon the class situation, the length of the class period, and the length of the school year.

Content. The title of the book indicates that the content provides an introduction to secondary school mathematics. Throughout the course emphasis is placed upon mathematics as a method of reasoning. The structure of our decimal numeration system is examined and then the counting numbers, whole numbers, rational numbers, and negative numbers are successively introduced.

The basic properties (field axioms) are intuitively developed as the successive sets of numbers are studied. The familiar computational procedures are shown to be legitimate because of the properties of the number system and the operations employed.

The number line and the idea of presenting numbers as points on a line provide the basis for all graphing and for analytical geometry. The number line provides the motivation for order relations between numbers and for the invention of real numbers.

Procedures for computing with decimal fractions are rationalized and percent is taught by means of proportion. Measurement is carefully developed, based on properties of continuous quantities.

The main purpose of the geometry included in this text is to present intuitively the concepts of point, line, and plane and to reach agreement by inductive reasoning that certain statements concerning these concepts appear to be true. Some of these statements will appear in the formal geometry course as axioms. Others will be proved as theorems. A second purpose of the geometry in this book is to present an introduction to the process of deductive reasoning in geometry.

Summary. We hope that by introducing the pupil to simple number theory, the development of the real number system, aspects of non-metric geometry, and the notions of ratio and proportion, in a carefully paced fashion which makes full use of a developmental approach, we shall be successful in attracting and retaining increased numbers of pupils for continued study of mathematics. We hope that appropriate mathematics, suitably taught, will awaken interest in pupils whose progress in traditional courses seemed hopeless. The discovery and nurture of heretofore unidentified capacity for learning mathematics is one of the main purposes of this book.

Chapter 1

WHAT IS MATHEMATICS?

General Remarks

This chapter is intended to give the pupil an appreciation of the importance of mathematics. Its objectives are:

I. To develop an understanding of what mathematics is; to dispel the notion that mathematics consists solely of computation.

II. To develop an appreciation of the role of mathematics in our culture.

III. To motivate pupils by pointing out the need for mathematicians and for mathematically trained people.

Since this chapter is much different from ordinary textbook material, it will need a different treatment. The purpose of the chapter is not to teach many facts or skills, but rather to build an enthusiasm for the study of mathematics. Good attitudes will be built if you use imagination and enthusiasm in getting these objectives across to the pupils.

It is expected that from five to six lessons will be sufficient for this chapter. Certainly no more than six days should be devoted to it.

It might be worthwhile to have the pupils read this chapter again at the end of the year. The problems might also be solved again. They should be much easier to solve after the course has been completed.

Encourage the more able students to solve the brainbusters but be ready to help them if they have difficulties. Most pupils will want to puzzle over the brainbusters for a few days. For this reason, only individual help is suggested until the time seems appropriate for general class discussion.

For the average and slower students, this chapter might present some real challenge. They will have to be led frequently to discover the solutions themselves. They should not be "given" the solutions, but enough hints should be given at the appropriate times that they will feel some degree of success at the beginning of their seventh grade course. These problems should be challenging, but also fun for the students. This is not a chapter to be tested.

1-1. Mathematics as a Method of Reasoning.

A class discussion might center around what the students think mathematics is. This could lead very well into Section 1-2 on Mathematical Reasoning.

1-2. Mathematical Reasoning.

It might provide an additional challenge to emphasize to the pupils that Exercises 1-2a and particularly 1-2b are not easy. Moreover, no simple formula for solution can be given. Some of the pupils (and many parents!) will certainly find the problems in 1-2b difficult and time-consuming at this stage. You may not wish to assign all the problems in this section.

Although there is no section on deductive reasoning, it is important to understand the distinction between inductive and deductive reasoning, and their applications to mathematics.

The experimental scientist arrives at a conjecture after a number of observations or trials in the laboratory. Further experimentation is used to prove or disprove the validity of this conjecture. This is inductive reasoning.*

The mathematician might also arrive at a conjecture by inductive reasoning. But he cannot prove the mathematical statement by experimentation. Knowing that a statement is true in a certain number of cases does not prove it true for all cases.

*Inductive reasoning should not be confused with mathematical induction which is a valid means of proof and depends upon a property of the counting numbers. See Haag, Vincent H., Studies in Mathematics, Vol. III: "Structure of Elementary Algebra".

A single case that contradicts the conjecture disproves it. The *proof* must depend on *deductive reasoning*; that is, it must be a statement which follows logically from a set of other statements which have been proved true or which are assumed true. Deductive reasoning is the "if-then" type of reasoning that mathematicians employ.

Answers to Questions in 1-2.

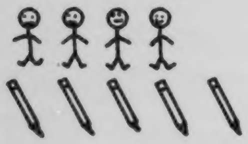
Yes, there can be empty boxes. If any pupils are born in the same month, there will be at least two slips in the same box and then at least one box will be empty. If all 12 pupils are born in the same month, then one box will have 12 slips and the other eleven will be empty. If all 12 pupils are born in different months, then each box will have exactly one slip in it.

When the 13th pupil places his slip, one box will have two slips. This idea is known as the pigeon-hole principle and is used as the basis for many mathematical proofs.

Answers to Exercises 1-2a

Since these are to be used for class discussion, plan simple demonstrations to illustrate them. The birthday illustration in the text may serve as a model for these problems.

1.



e.g., One of the students will get this 5th pencil.

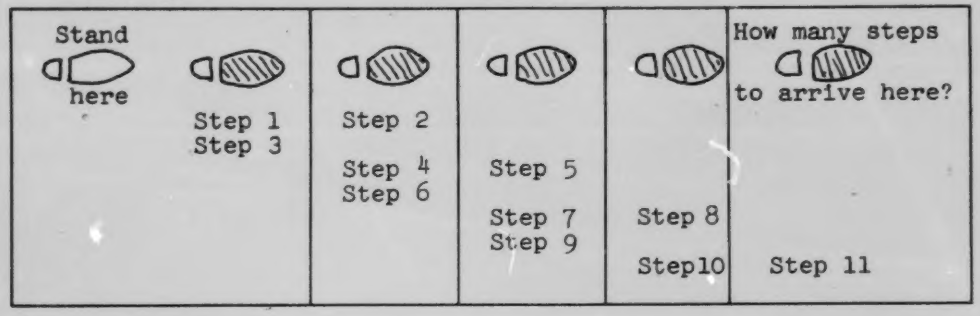
2. Illustrate possibilities as in the birthday problem discussion. Ask, "Is it possible for one or more students to receive only one pencil?" Then illustrate the case with two pencils for each student. Discuss what happens to the 13th and 14th pencils.

3. a. 8
   b. 15
   c. (a) If there are 7 movie houses in a town, then 8 is the smallest number of people that would be required to go to the movies to be sure that at least 2 people see the same show.
      (b) If there are 7 movie houses in a town, then 15 is the smallest number of people who must go to the movies to be sure that at least 3 people see the same show.

Exercises 1-2b which follow are usually difficult for the students, but if they are not pushed into finding a solution quickly, they should enjoy them. The problem involving the wolf, goat and the cabbage should be discussed thoroughly in class, so the students will see the pattern of the problem. There are many problems of this sort. In this particular one, the goat is the key to the solution, since he is the only one who cannot be left unguarded with either the wolf or the cabbage. In the class discussion, the class might be asked if there might be another solution. Actually, the wolf and cabbage can be interchanged in the solution.

Answers to Exercises 1-2b

1. 11 steps. With the first three steps (two forward and one backward) she progresses one space ahead. She does this 3 times (9 steps) and is at that time 3 spaces ahead. Two more steps and she is in the pool.



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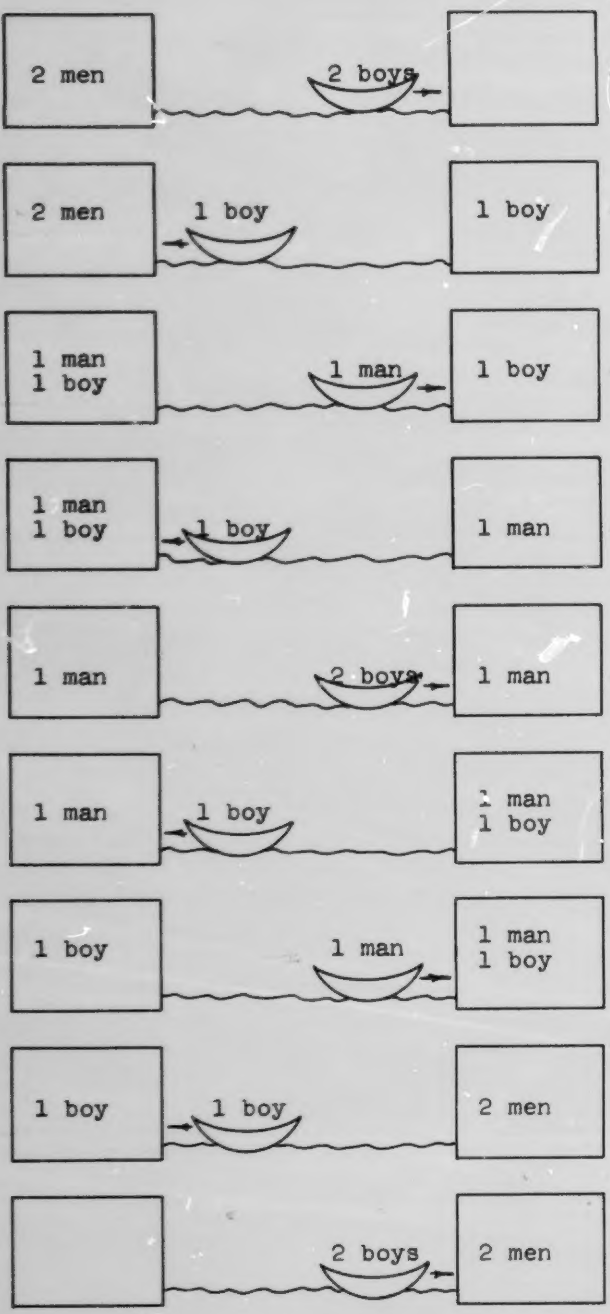
2.


2 men
2 boys
2 men
1 boy
1 boy
1 man
1 boy
1 man
1 boy
1 man
1 boy
1 boy
1 man
1 man
2 boys
1 man
1 man
1 boy
1 man
1 boy
1 boy
1 man
1 man
1 boy
1 boy
1 boy
2 men
2 boys
2 men

-(A) Two boys go over.

-(B) One boy comes back.

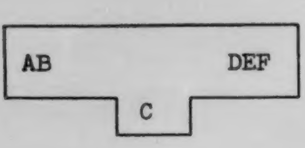
-(C) One man goes over.

-(D) The other boy comes back

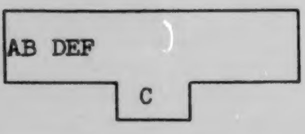
-(E) Steps A-D are repeated in order to bring over the other man

page 4: 1-2

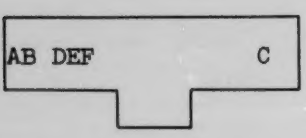
3.



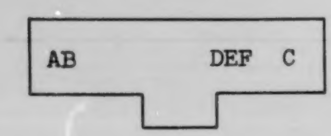
--Step 1: Boat C enters bay G.



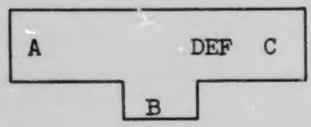
--Step 2: Boats DEF pass G.



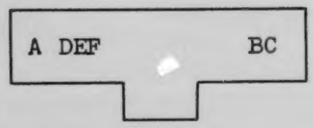
--Step 3: Boat C goes on its way.

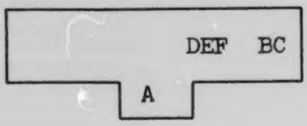


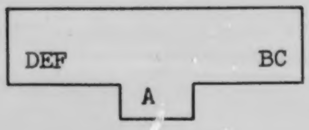
--Step 4: Boat DEF returns to original side of G.

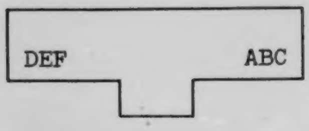


--Step 5-9: The same operation (Steps 1-4) is repeated for Boat B, and then for Boat A.









4. Two sons cross; one returns. Father crosses; other son returns. Two sons cross.

5. Man takes goose and returns alone. He takes fox and returns with goose. He takes corn across river and returns alone to pick up goose. (Does the class see that this one is identical in structure to the illustrative problem?)

The following problems will be done by the class by trial-and-error, probably. The explanations of these are of interest to teachers, however, since these are solutions of Diophantine equations. (The idea in Diophantine equations is to find integer solutions.) These three problems are given in order of increasing difficulty. The algebraic discussion is for the teacher, not the pupil.

6. Yes. This depends on the fact that $8x + 5y = 3$ has solutions in integers. One solution is, $x = 1$, $y = -1$. This means that if you fill the 8 gallon jug once and pour it into the 5 gallon jug just once, you will have 3 gallons left in the 8 gallon jug.

7. Yes. This depends on the fact that $8x + 5y = 2$ has solutions in integers, such as $x = -1$, $y = 2$ and $x = 4$, $y = -6$. The first means that if you fill the 5 gallon jug twice and empty it once into the 8 gallon jug, you will have 2 gallons left in the 5 gallon jug. The second solution means that if you fill the 8 gallon jug four times and use it to fill the 5 gallon jug 6 times, you will have 2 gallons left in the 8 gallon jug. Point out that the first solution is better.

*8. Yes. This depends on the fact that $8x + 5y = 1$ has solutions in integers, such as $x = 2$, $y = -3$, and $x = -3$, $y = 5$. The first means that if you fill the 8 gallon jug twice and empty it into the 5 gallon three times, you will have one gallon left in the 8 gallon jug. The second solution means that if you fill the 5 gallon jug 5 times, and empty it into the 8 gallon jug 3 times, you will have one gallon left in the 5 gallon jug. Point out that the first solution is best.

1-3. From Arithmetic to Mathematics.

John Friedrich Karl Gauss was born in Brunswick, Germany, in 1777. He died in 1855 at the age of 78. The pupils may be interested in noting that his lifetime almost spanned the years from the American Revolution to the Civil War.

Mathematicians consider Gauss as one of the greatest mathematicians of all times.

In this age of space exploration it is interesting to note that Gauss developed powerful methods of calculating orbits of comets and planets. His interests extended also to such fields as magnetism, gravitation, and mapping. In 1833 Gauss invented the electric telegraph, which he and his fellow worker, Wilhelm Weber, used as a matter of course in sending messages.

In 1807 Gauss was appointed Director of the Göttingen Observatory and Lecturer of Mathematics at Göttingen University. In later years the greatest honor that a German mathematician could have was to be appointed to the professorship which Gauss once held.

This section deals with Gauss's discovery of the common method of summing an arithmetic series. It dramatizes how some pupils (and mathematicians) apply insight to finding a solution to a problem. Your better students should be told that there are methods other than Gauss's for finding the sum of a series of numbers. Some students might be encouraged to discover methods of their own for adding number series quickly.

The "middle number" method is one that may be used. This scheme can be used for an even or an odd number of integers. The following examples may be used to explain this method to the students who have tried to discover other methods.

Example A. $1 + 2 + 3 + 4 + 5 + 6 + 7 = ?$

In this series the middle number (4) is the average of the individual numbers of the series. The sum is the product of the middle number (4) and the number of integers in the series, or $4 \times 7 = 28$.

Some pupils may prefer to think of the series as

$$\begin{aligned}&(1 + 7) + (2 + 6) + (3 + 5) + 4 = \\ &(4 + 4) + (4 + 4) + (4 + 4) + 4 = \\ &\qquad\qquad 7 \times 4 = 28.\end{aligned}$$

Example B. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = ?$

In this case the "middle number" is half way between 4 and 5, or $4\frac{1}{2}$. Then the product $(4\frac{1}{2}) \times 8 = 36$ is seen to give the correct sum.

It may seem more plausible here to write the sum as

$$(1 + 8) + (2 + 7) + (3 + 6) + (4 + 5) =$$

$$(4\tfrac{1}{2} + 4\tfrac{1}{2}) + (4\tfrac{1}{2} + 4\tfrac{1}{2}) + (4\tfrac{1}{2} + 4\tfrac{1}{2}) + (4\tfrac{1}{2} + 4\tfrac{1}{2}) = 8 \times 4\tfrac{1}{2}$$

Clearly, Gauss's method is to be preferred in this case.

Answers to Exercises 1-3

1. 15. Another method is this: $2 + 4 = 3 + 3$, $1 + 5 = 3 + 3$. That is, the sum is the same as: $3 + 3 + 3 + 3 + 3 = 5 \times 3 = 15$. This can be called the "averaging method".

2. Either method works. Gauss method: $\frac{8 \times 5}{2}$;

   Averaging method: $5 \times 4 = 20$.

3. $\frac{16 \times 8}{2} = 64$. Here there is an even number of quantities so that the "averaging method" must be modified to give 8 eight's or $8 \times 8 = 64$.

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4. a. 4 4

b. 9 9

c. 16 16

d. They are the same.

This is really the "averaging method." See Problem 1.

$$\begin{aligned} 1+3 &= 2+2 &&= 4 = 2\times 2\\ 1+3+5 &= 3+3+3 &&= 9 = 3\times 3\\ 1+3+5+7 &= 4+4+4+4 &&= 16 = 4\times 4 \end{aligned}$$

e. $1 + 15 = 3 + 13 = 5 + 11 = 7 + 9 = 16$

The average of each two is 8.

Therefore $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 =$

$(1 + 15) + (3 + 13) + (5 + 11) + (7 + 9) =$

$8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 = 8 \times 8 = 64$

5. $$\begin{array}{r} 7 + 9 + 11 + 13 + 15 + 17\\ \underline{17} + \underline{15} + \underline{13} + \underline{11} + \underline{9} + \underline{7}\\ 24 + 24 + 24 + 24 + 24 + 24 \end{array} \qquad \frac{24\times 6}{2} = 72$$

6. $$\begin{array}{r} 4 + 6 + 8 + \ldots + 28\\ \underline{28} + \underline{26} + \underline{24} + \ldots + \underline{4}\\ 32 + 32 + 32 + \ldots + 32 \end{array} \qquad \frac{32\times 13}{2} = 208$$

7. $$\begin{array}{r} 1 + 2 + 3 + \ldots + 200\\ \underline{200} + \underline{199} + \underline{198} + \ldots + \underline{1}\\ 201 + 201 + 201 + \ldots + 201 \end{array} \qquad \frac{200\times 201}{2} = 20{,}100$$

$$\begin{array}{r} 0 + 1 + 2 + 3 + \ldots + 200\\ \underline{200} + \underline{199} + \underline{198} + \underline{197} + \ldots + \underline{0}\\ 200 + 200 + 200 + 200 + \ldots + 200 \end{array} \qquad \frac{201\times 200}{2} = 20{,}100$$

Yes, the answers are the same. If we start with 1, there are 200 integers in the series giving us $\frac{(1 + 200)200}{2}$.

If we start with 0, there are 201 integers in the series, giving us $\frac{(0 + 200)201}{2}$.

The products of the same factors are equal. The method also may be used in a series if we select a number other than 1 or 0 as the starting points. Some of the better students may investigate whether the method works in other series.

1-4. Kinds of Mathematics.

Discussion of this section should emphasize the dynamic character of mathematics. It is not a "dead" subject as many parents believe.

It is important also to point out here (and throughout the course) that certain important ingredients are common to all the many varieties of mathematics. The method of logical reasoning, the use and manipulation of abstract symbols, the insistence on precision of thought and clarity of expression, the emphasis on general results--these are some characteristics which need to be stressed whenever possible.

Some Interesting Problems

Bring out the fascination of mathematics as a leisure activity or hobby. Encourage students to look for recreational mathematics in books available at school and from current magazines or rotogravure sections of newspapers.

Choose such problems now and then, throughout the year, at a time when the class needs a change of pace. These kinds of problems can be used profitably with the class period before a lengthy vacation.

Some Recreational Books

Adler, Irving. MAGIC HOUSE OF NUMBERS. New York: Signet Key Book, the New American Library, Inc. Paperbound 50¢.

Adler, Irving. MATHEMATICS, THE STORY OF NUMBERS, SYMBOLS, AND SPACE. New York: Golden Press, 1958. Library edition $1.39. Paperbound 50¢.

Kraitchik, Maurice. MATHEMATICAL RECREATIONS. New York: Dover Publications, Inc., 1953. Paperbound $1.75.

Leeming, Joseph. MORE FUN WITH PUZZLES. New York: J. B. Lippencott Co., 1947. $2.50.

Merrill, Helen A. MATHEMATICAL EXCURSIONS. New York: Dover Publications, 1957. Paperbound $1.00.

Mott-Smith, Geoffrey. MATHEMATICAL PUZZLES FOR BEGINNERS AND ENTHUSIASTS. New York: Dover Publications, Inc., 1954. Paperbound $1.00.

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Smith, David Eugene and Ginsburg, J. NUMBERS AND NUMERALS. Washington, D.C.: National Council of Teachers of Mathematics. 35¢.

Wylie, C. R. 101 PUZZLES IN THOUGHT AND LOGIC. New York: Dover publications, Inc., 1957. Paperbound $1.00.

Answers to Exercises 1-4

1. Cost: $80 + $75 = $155

   Selling price: $90 + $100 = $190

   Profit = Selling price - Cost = $190 - $155 = $35.

2. There are five combinations:
   - 1 quarter and 1 nickel
   - 1 dime and 4 nickels
   - 2 dimes and 2 nickels
   - 3 dimes
   - 6 nickels

3. a. 6 c. None e. 12 g. 1
   b. 27 d. 8 f. 6

4. There is, of course, usually more than one way to do these.

$$4 = \frac{4 - 4}{4} + 4 \quad \text{or } 4(4 - 4) + 4$$

$$5 = \frac{(4 \times 4) + 4}{4}$$

$$6 = \frac{4 + 4}{4} + 4$$

$$7 = \frac{44}{4} - 4$$

$$8 = 4 + 4 + 4 - 4$$

$$9 = 4 + 4 + \frac{4}{4}$$

$$10 = \frac{44 - 4}{4}$$

If the students like these, you might do a few extra ones such as the ones below; however, you can not go too high or have too great a variety since the students are not familiar enough with exponents. Some other examples are as follows:

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$$12 = \frac{44 + 4}{4}$$

$$15 = \frac{44}{4} + 4 \quad \text{or} \quad (4 \times 4) - \frac{4}{4}$$

$$16 = \frac{4 \times 4 \times 4}{4}$$

$$17 = (4 \times 4) + \frac{4}{4}$$

$$20 = 4(4 + \frac{4}{4})$$

$$24 = (4 \times 4) + 4 + 4$$

$$28 = 44 - 4 \times 4$$

$$32 = 4 \times 4 + 4 \times 4$$

$$43 = 44 - \frac{4}{4}$$

$$44 = 44(\frac{4}{4})$$

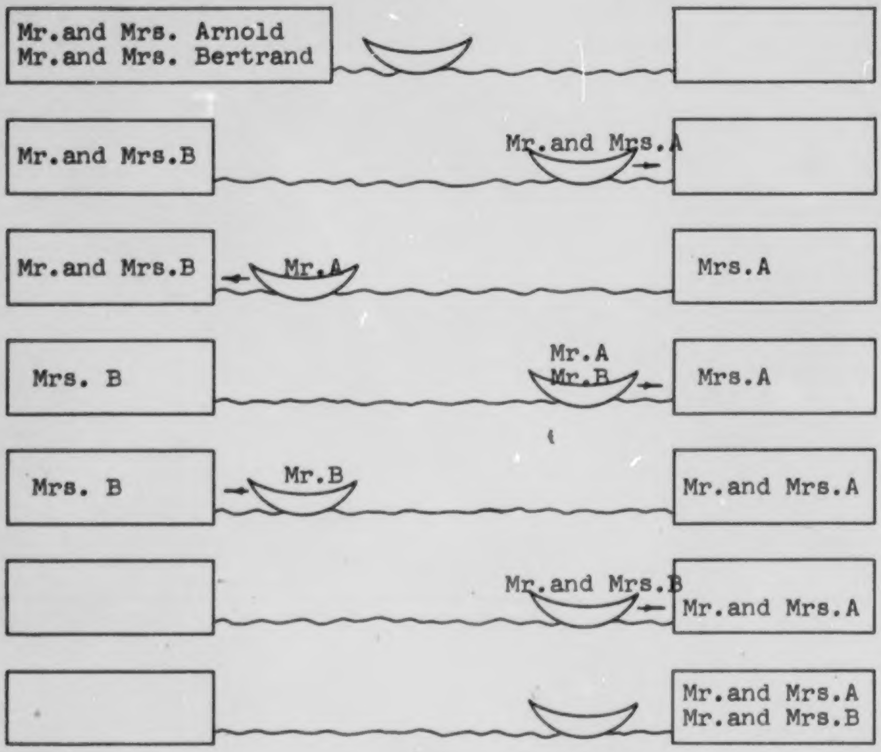
$$45 = 44 + \frac{4}{4}$$

$$60 = 4 \times 4 \times 4 - 4$$

$$68 = 4 \times 4 \times 4 + 4$$

$$256 = 4 \times 4 \times 4 \times 4$$

5. One solution would be as follows:


Mr.and Mrs. Arnold
Mr.and Mrs. Bertrand
Mr.and Mrs.B
Mr.and Mrs.A
Mr.and Mrs.B
Mr.A
Mrs.A
Mrs. B
Mr.A
Mr.B
Mrs.A
Mrs. B
Mr.B
Mr.and Mrs.A
Mr.and Mrs.B
Mr.and Mrs.A
Mr.and Mrs.A
Mr.and Mrs.B

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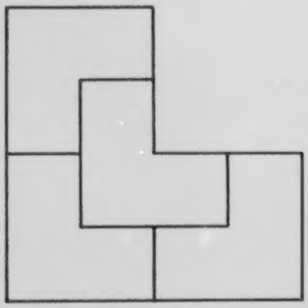
6.



7. Ask if it is practical to try out all the possible ways the dominoes may be placed on the board. This would be difficult because there are many thousands of ways of trying to do this. The solution may be found in another way:

> There are 64 squares in all, and 62 squares to cover. If 31 dominoes are placed on the board, you must cover 31 white and 31 black squares. However, since both of the squares not to be covered are black there are therefore 32 white squares, but only 30 black squares to be covered. This is evidently impossible, because each domino always covers both a white square and a black square.

Be sure the pupils do not confuse the notion of an unsolved problem with that of an impossible problem.

*1-5. Mathematical Discovery.

Just as music is the art of creating beauty with sounds, and painting is the art of creating beauty with colors and shapes, so mathematics is the art of creating beauty with combinations of ideas. Many people enjoy mathematics as a fascinating hobby. Many people study mathematics for fun as other people enjoy music or painting for pleasure.

The problems on tracing are designed to set a pattern of thinking which can be utilized directly in showing the impossibility of the Königsberg problem.

Answers to Exercises 1-5

1. Figure 5, figure 6, figure 7, figure 8, figure 10, and figure 12 can be retraced.

2.

FIGURE	A	B	C	D	E	F	G	H	I	NUMBER OF POINTS WITH ODD NUMBER OF PATHS	TRACEABLE YES OR NO
1	2	2	4	2	2					0	Yes
2	3	2	3	2						2	Yes
3	3	3	2	3	3					4	No
4	2	3	3	4	3	2	3			4	No
5	2	2	2							0	Yes
6	2	3	3	2						2	Yes
7	2	2	4	2	2					0	Yes
8	3	3	2	2	4					2	Yes
9	3	3	3	3	4					4	No
10	4	2	4	3	3	4				2	Yes
11	3	3	3	3	3	3				6	No
12	4	2	4	4	4	4	2			0	Yes
13	5	2	5	5	5	4	2	2	2	4	No

3. A figure is traceable if there are at most two odd vertices. A figure is not traceable if there are more than two odd vertices.

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If the students ask for a more detailed explanation, the teacher may use the following approach.

In the preceding exercises you found that the figures having more than two points with an odd number of paths were not traceable. Let us try to see why this is so.

A point that has an even number of paths presents no special problem in tracing. For example, in figure 1 with two paths through point A, we can move in to the point on one path and out of the point on the other path. Also, in figure 2, with four paths we can move in to point B on the first path, out of point B on the second path, in to point B on the third path, and out of point B on the fourth path. A similar procedure can be followed with any point having an even number of paths.

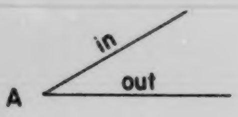

FIGURE 1

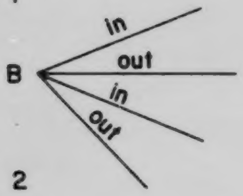

FIGURE 2

A point that has an odd number of paths presents a different situation.

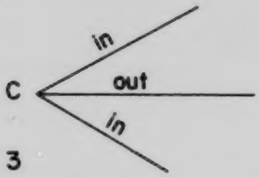

FIGURE 3

In figure 3, with three paths, we can move in to the point on one path, and in to the point on the third path. In this case, the drawing must end at the point since we have no second outgoing path.

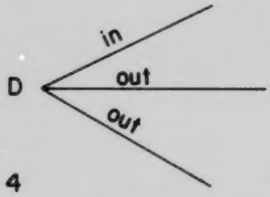

FIGURE 4

In figure 4, with three paths, we can move out of the point on one path and in to the point on the second path. However, the third (out) path must be used to leave again. Thus, the point must be a starting point.

We have shown that a point that has an odd number of paths must be either a starting point or an ending point. Since a figure can have only one starting point and one ending point, a figure having more than two points with an odd number of paths is not traceable.

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*1-6. The Königsberg Bridge Problem.

After the completion of Section 1-5 the Königsberg Bridge Problem should not be very difficult for pupils to follow. They should be able to explain why the problem is impossible. Figure 1-6b will help them to see the many different ways of walking through the city using the bridges to go from one piece of land to another. Use C in place of the piece of land to the north and D, the land to the south. A is the island and B is the land to the east. Think of each piece of land as shrinking to a point. This does not change the problem. The lines leading from A, C, D, and B show routes across the bridges to the various parts of the city. At points B, C, and D three routes come together and at point A five routes meet. Since there are four points where an odd number of routes come together, it is impossible to walk over each bridge once and only once.

The proof of this problem was derived by Leonhard Euler.

1-7. Mathematics Today.

Students should be encouraged throughout the year to bring material concerning mathematics for the bulletin board. There is a wealth of material in daily papers, magazines and pamphlets. Even in the want-ads of large daily papers there are advertisements for mathematics.

Before World War II almost all mathematicians were employed as teachers in schools and colleges. Since then, the world of mathematics and the world of mathematicians have changed tremendously. Today there are more teachers of mathematics than ever before. In junior and senior high schools there are perhaps

50,000 who teach mathematics. Employed in colleges and universities there are about 3,000 more, but now in business, in industry, and in government there are from 7,000 to 10,000 persons working as mathematicians.

Numerous agencies of the Federal Government hire mathematicians for a number of different assignments. Literally, thousands of people work with computers and computer mathematics for the big electronic computers. Industries of all types are hiring mathematicians to solve complex mathematical problems, to help other workers with mathematical difficulties, and even to teach mathematics to other employees.

These changes have been brought about by the revolutionary advances in science and in technology which we discussed. Changes continue to take place.

Many people who are not primarily mathematicians need a comprehensive background in mathematics. This has long been true of engineers and physicists, and they now find it necessary to use even more advanced mathematics. Every new project in aircraft, in space travel, or in electronics demands greater skills from the engineers, scientists, and technicians.

A survey of college requirements in certain vocations might be interesting for the class but the necessity for a minimum knowledge for everyone should also be stressed.

The highlights of junior high mathematics should be discussed with the class. During this year the students will develop a better understanding of what mathematics really is. They will have many opportunities to use mathematical reasoning. Though mathematics is much more than just counting, computing, measuring and drawing, many operations and applications will be used in the following chapters.

They will learn about the history of numbers from the primitive peoples' scratches in the dirt, to written symbols for numbers. Early number symbols are reviewed to emphasize the characteristics of the numeration system we use. Pupils will find that the numeral 100 (read one, zero, zero) does not always represent one hundred.

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For many years they have used counting numbers, such as 1, 2, 3, 4, and so on. Are there other kinds of numbers? Yes, they will become acquainted with several other kinds.

Ask them if they have ever carefully observed how numbers behave when you add or multiply them. If they have, they will find some properties that are always true in addition and multiplication. Zero and one also have special properties which they may have discovered. This year they will observe numerals much more closely than they have ever done. For some of them it will be similar to looking through a magnifying glass. When they really look at a problem carefully, they discover how much clearer the mathematics in the problem becomes.

For many years they have used the word "equal" and know a symbol for it. Can there be inequalities as well?

Another interesting part of their year will be spent considering ideas of point, line, plane, and space. They may already have some ideas about these. Have they ever built models? If they have, they will have some of their own ideas of point, line, plane, and space.

The students are already familiar with many symbols in mathematics. Some of these symbols have been used so often that they are used without thinking much about them. Look at the fraction $\frac{23}{30}$. Are they familiar with this symbol? Now look at an Egyptian way of writing this fraction many years ago: $\frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{15}$. Will they not find the symbol $\frac{23}{30}$ much simpler and easier to handle than the sum of these four fractions?

New symbols will be introduced this year to enable the students to be more precise mathematically.

You cannot possibly tell them all about their first year in junior high mathematics, or what mathematics is, at the end of just one chapter. However, it is hoped that as they study mathematics this year they will gain a much better idea of what mathematics is and why they should know as much of it as they can learn.

Bibliography for the Teacher

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2. Boehm, George A. W. and the Editors of Fortune. THE NEW WORLD OF MATHEMATICS. New York: The Dial Press, 1959.
   See especially pp. 17-46 (The New Mathematics).
3. Courant, Richard and Robbins, Herbert. WHAT IS MATHEMATICS? New York: Oxford University Press, 1943.
4. Kline, Morris. MATHEMATICS IN WESTERN CULTURE. New York: Oxford University Press, 1953.
   See especially pp. 3-12 (General Concepts),
   pp.21-39 (Deductive Reasoning),
5. Sawyer, W. W. PRELUDE TO MATHEMATICS. Baltimore, Maryland: Penguin Books, 1957.

Chapter 2

NUMBER SYMBOLS

Introduction

For this unit little background is needed except familiarity with the number symbols and the basic operations with numbers. The purpose of the unit is to deepen the pupil's understanding of the decimal notation for whole numbers, especially with regard to place value; this will help him delve a little deeper into the reasons for the operations, which he already knows, for addition and multiplication. One of the best ways to accomplish this is to consider systems of number notations using bases other than ten. Since, in using a new base, the pupil must necessarily look at the reasons for "carrying" and the other mechanical operations in a new light, he should gain deeper insight into the decimal system. A certain amount of computation in other systems is necessary to fix these ideas, but such computation *should not* be regarded as an end in itself. Some of the pupils, however, may enjoy developing a certain proficiency in using new bases in computing.

The most important reason for introducing ancient symbolisms for numbers is to contrast them with our decimal system, in which not only the symbol, but its position, has significance. It should be shown, as other systems are presented, that position has some significance in them also. The Roman system made a start in this direction in that XL represents a different number from LX, but the start was a very primitive one. The Babylonians also made use of position, but lacked a symbol for zero until about 200 B.C. The symbol "Σ" denoted the absence of a figure but apparently was not used in computation. The numeral zero is necessary in a positional system. Pupils should *not* be expected to memorize ancient symbolism. It is recommended that very little time be spent on the use of the symbols themselves. In order for pupils to appreciate the important characteristics of our system of writing numerals, the following table may be discussed.

	BASE	PLACE VALUE	ZERO
Egyptian	Ten	No	No
Babylonian	Sixty	Yes	limited meaning
Roman	Varied	No (but it has positional value)	No
Decimal	Ten	Yes	Yes

It is especially important to distinguish between a number and the symbols by which it is represented. Some of the properties usually connected with a number are really properties of its notation. The facts that, in decimal notation, the numeral for a number divisible by 5 ends in 5 or 0, and that $\frac{1}{3}$ has an unending decimal equivalent are illustrations. Most of the properties with which we deal are properties of the numbers themselves, and are entirely independent of the notation in which they are represented. Examples of such properties of numbers are: $2 + 3 = 3 + 2$; the number eleven is a prime number; and six is greater than five. The distinction between a number and the notation in which it is expressed should be emphasized whenever there is opportunity.

An attempt has been made to use "number" and "numeral" with precise meaning in the text. For example, "numerals" are written, but "numbers" are added. A numeral is a written symbol. A number is a concept. Later in the text it may be cumbersome to the point of annoyance to speak of "adding the numbers represented by the numerals written below." In this case the expression "adding the numbers below" may be used.

At several points, numbers are represented by collections of x's. Exercises of this kind are important, because they show the role of the base in grouping the x's, as well as the significance of the digits in the numeral for the number.

Suggested Time Schedule

It is important that only enough time be spent on the various sections to secure the understandings desired. The historical symbols themselves are not important. Neither are the numerals in other bases valuable in themselves, but the ideas that they help to clarify are important. It is estimated that 22 to 25 days will be required for completion of this chapter, including testing. A few days more or less may be required, depending upon the character of the particular class.

Familiarity with the subject matter is an important factor in a smooth presentation. Teachers report that a second experience with this material is much easier than the first. The lesson moves more rapidly, apparently, as the teacher gains confidence in the subject matter presented.

Homogeneously grouped classes undoubtedly will alter the suggested schedule since the more able students can complete the chapter in less time while less able students may require a considerably longer period of time on various sections. The following schedule may then be adapted to local needs, taking into consideration the length of class periods, and other factors. It should be remembered that extra time spent on this chapter will necessarily reduce the number of days available for later important chapters.

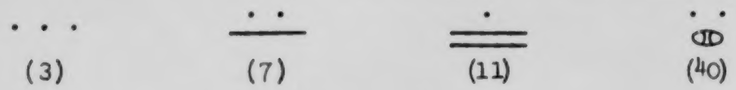
Sections	Days
1	2
2	2 or 3
3	3
4	4
5	3
6	1
7	1 or 2
8	1
9	3
10	1
Test	1
Total	22 to 24

2-1. Ancient Number Symbols.

The purpose of the historical material is to trace the continuing need for convenient symbols and for a useful way of writing expressions for numbers. The idea of "one-to-one" correspondence is introduced. It is developed later and should not be defined here where the emphasis is upon numerals rather than upon numbers. Egyptian symbolism is introduced to familiarize the pupils with one of the first important systems of notation. Do not use an excessive amount of time in discussing the Egyptian or Babylonian systems. No pupil should be required to memorize ancient symbolism except in the case of Roman numerals.

The Babylonians were among the first to use place value. They used the two symbols Y and ≺ for small numbers and gave them a positional value using base sixty. An empty space had somewhat the same meaning as our zero. For example, Y Y could mean (60 × 1) + 1. Similarly, ≺YY could mean (60 × 10) + 2. On the other hand, the first illustration could be interpreted as (60 × 60) + 1, or even as 1 + 1, since there is no means of determining how much space the writer considered the equivalent of one place. The indefinite means of indicating position, and the base of 60 makes it a difficult system to understand. For this reason, it was touched lightly in the student text. A more complete explanation may well be given by the teacher if he wishes. It can be pointed out that our measurement of angles and of time is a heritage from the Babylonians.

The Mayan numerals illustrate another method by which a multiplier in a numeral system can be indicated. The Mayans used three symbols: a dot for 1, a bar for five, and an oval, ↀ , which written below another symbol multiplied its value by twenty. Some Mayan numerals are written below:



Hogben's "Wonderful World of Mathematics" contains a very attractive account of Mayan and other numerals.

The Roman system may be stressed because of its continued use. Note that the subtracting principle was a late development. It may be pointed out that computation in ancient symbolism was complex and sometimes very difficult. Because of this, various devices were used, such as the sand reckoner, counting table, and abacus. After decimal numerals became known, algorithms were devised and people were able to calculate with symbols alone. There was much opposition in Europe to the introduction and use of Hindu-Arabic numerals, especially on the part of the abacists. As the new system became accepted, the abacus and other computing devices slowly disappeared in Europe.

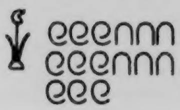
Answers to Exercises 2-1a

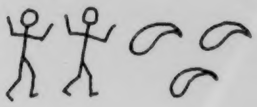
1. a. |||

   b. ∩||

   c. ∩∩∩

   d. ℯℯ∩∩ ||| ||

   e. 

   f. 

2. a. 32 b. 214 c. 340 d. 1,250

3. ∩∩∩|| or ||∩∩∩ or ∩∩|∩| or ∩∩||∩ , etc.

4. a. 7, ||| ||| |

   c. 204, ℯℯ || ||

   b. 15, ∩ ||| ||

   d. 10,351 ℯℯℯ ∩∩∩ | ∩∩

Other representations should be accepted: for example, 𝍸 || for seven, etc.

Answers to Exercises 2-1b

1. a. YYYY b. <Y c. <<YYY d. <YYY
<YYY
<YY

2. a. 15 b. 37 c. 55

3. a. 16 c. 29 e. 90 g. 666
b. 14 d. 110 f. 105 h. 2350

4. a. XV e. XCVIII
b. XXIII f. DCXXIX
c. XXXIV g. MMMCCLVI
d. LXII

5.

	Decimal	Roman	Egyptian	Babylonian
a.	6	VI	||| |||	YYY YYY
b.	17	XVII	∩||| ||| |	YYY <YYY Y
c.	24	XXIV	∩∩|| ||	<YY <YY

6. a. 7
b. 1
c. 7
d. 10

2-2. The Decimal System.

Discussion of the decimal system should emphasize the importance of the invention of a useful system which lends itself easily to calculation. Its efficiency lies in the small number of symbols used, with no need for new or additional symbols as larger numbers are introduced; in the use of place value where each position corresponds to a power of the base; in the use of zero as a place holder and in computation. The students' appreciation for some of these characteristics will be increased as they proceed through Section 2-3. Emphasize the value represented by a digit and the value of position in decimal notation.

An abacus can be used to good advantage in discussion of place value. It is suggested that counting be done as a class exercise, each number shown on the abacus, and similarities between this representation and numerals discussed. It can be pointed out that the Romans used the idea of place value in their computation with an abacus, but did not extend the principle to numerals. The invention of a zero symbol might have changed the course of Roman arithmetic.

The amount of attention given to reading and writing of numerals will depend upon the needs of the pupils. Some pupils probably will have a very limited proficiency in this area.

Students should be aware of the meanings of number names such as thirteen, (three and ten), seventeen (seven and ten), forty (four tens), sixty (six tens), etc.

Children will enjoy stories that illustrate the difficulties involved in trying to discard outmoded systems of record keeping and in learning to compute with the decimal system.

For a long period in English history, exchequer accounts were kept by means of wood tallies notched to show amounts. Notches of different sizes represented different amounts of money. Not until 1826 was the practice finally abolished. The following quotation from an address by Charles Dickens, delivered a few years later, describes the official end of the era.

> "In 1834 it was found that there was a considerable accumulation of [these tallies]; and the question then arose, what was to be done with such worn-out, worm-eaten rotten old bits of wood? The sticks were housed in Westminster, and it would naturally occur to any intelligent person that nothing could be easier than to allow them to be carried away for firewood by the miserable people who lived in that neighborhood. However, they never had been useful, and official routine required that they should never be, and so the order went

> out that they were to be privately and confidentially burned. It came to pass that they were burned in a stove in the House of Lords. The stove, over-gorged with these preposterous sticks, set fire to the panelling; the panelling set fire to the House of Commons; the two houses were reduced to ashes; architects were called in to build others; and we are now in the second million of the cost thereof."

A story is told of a German merchant, living in the fifteenth century, who wished to give his son an advanced commercial education. He asked a prominent professor of a university to advise him as to where the son should be sent. The reply was that if training in addition and subtraction were sufficient, it could probably be obtained in a German university; but for instruction in multiplication and division, the son must be sent to Italy, where scholars had made considerable study of the art.

Answers to Exercises 2-2a

1. a. three hundred
   b. three thousand, five
   c. seven thousand, one hundred, nine
   d. fifteen thousand, fifteen
   e. two hundred thirty-four thousand
   f. six hundred eight thousand, fourteen
   g. one hundred thousand, nine
   h. one million, twenty-four thousand, three hundred five
   (Note: Only the tens numbers are hyphenated, as twenty-three.)
   i. thirty million, two hundred fifty thousand, eighty-nine
   j. fifty-two billion, three hundred sixty million, two hundred fifteen thousand, seven hundred twenty-three

2. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

3. thousand

4. a. 4 b. 2 c. 1 d. 3 e. 5 f. 5

5. a. Answers will vary.
   b. Answers will vary.

6. a. 100,000
   b. one hundred thousand
   c. Answers will vary.

7. a. 999,999
   b. nine hundred ninety-nine thousand, nine hundred ninety-nine.

8. a. 159
   b. 502
   c. 5,200
   d. 6,857
   e. 27,017
   f. 111,000
   g. 3,003,003
   h. 5,000,000,002
   i. 2100
   j. 6000

9. a. 857 b. 333,000 c. 910 d. 330,000

10. a. 10
    b. 10
    c. 10
    d. 10
    e. 100
    f. 1000
    g. 10
    h. 1000

11. 10 times as large.

Note the phrase "as large as" in the statement of the problem. Encourage the use of this phrase rather than "how many times larger than", which contains two conflicting ideas.

Exercises 2-2b were included as review material for maintenance of skills. Emphasize that knowledge of a procedure in calculation is of little value unless it leads to the correct result.

Answers to Exercises 2-2b

1. 135
2. 254
3. 1858
4. 35
5. 278
6. 605
7. 3948
8. 15,466
9. 334,100
10. 3,276,000
11. 124
12. 107
13. 600
14. 304 R 4
15. 4030 R 8.

2-3. Expanded Form and Exponents.

Exponents are introduced here in a situation which shows clearly their usefulness for concise notation. Furthermore, their use serves to emphasize the role of the base and of position. This role will be more fully utilized in the sections to follow. Note the use of parentheses to show that certain combinations are to be considered as representing a single numeral.

Use of the terms "square" and "cube" in reading second and third powers of numerals should not be introduced here.

Answers to Exercises 2-3a

1. 432, 234, 38
2. 2380, 300, 60,385
3. 3456, 402, 56,420, hundred
4. a. $28 = (2 \times 10) + (8 \times 1)$
   b. $56 = (5 \times 10) + (6 \times 1)$
   c. $721 = (7 \times 100) + (2 \times 10) + (1 \times 1)$
      or $[(7 \times 10 \times 10) + (2 \times 10) + (1 \times 1)]$
   d. $1312 = (1 \times 1000) + (3 \times 100) + (1 \times 10) + (2 \times 1)$
      or $[(1 \times 10 \times 10 \times 10) + (3 \times 10 \times 10) + (1 \times 10) + (2 \times 1)]$
   e. $244 = (2 \times 100) + (4 \times 10) + (4 \times 1)$
   f. $2846 = (2 \times 1000) + (8 \times 100) + (4 \times 10) + (6 \times 1)$
   g. $507 = (5 \times 100) + (0 \times 10) + (7 \times 1)$
   h. $23{,}162 = (2 \times 10{,}000) + (3 \times 1000) + (1 \times 100) + (6 \times 10) + (2 \times 1)$

Since our system has base ten, multiplying and dividing by powers of ten can be accomplished easily by changing the place value. These exercises give the teacher a chance to see if everyone in the class realizes this.

Answers to Oral Exercises 2-3b

1. 3,040
2. 304,000
3. 27,500
4. 22,200
5. 600
6. 99,000
7. 14,000
8. 45,000
9. 48,000
10. 64,000

Divide:

11. 270
12. 27
13. 27
14. 305
15. 100
16. 330
17. 11
18. 1,000
19. 10
20. 1

Answers to Exercises 2-3c

1. $423 = (4 \times 10 \times 10) + (2 \times 10) + (3 \times 1)$
2. $771 = (7 \times 10 \times 10) + (7 \times 10) + (1 \times 1)$
3. $5253 = (5 \times 10 \times 10 \times 10) + (2 \times 10 \times 10) + (5 \times 10) + (3 \times 1)$
4. $2608 = (2 \times 10 \times 10 \times 10) + (6 \times 10 \times 10) + (0 \times 10) + (8 \times 1)$

*5. $34{,}359 = (3 \times 10 \times 10 \times 10 \times 10) + (4 \times 10 \times 10 \times 10) + (3 \times 10 \times 10) + (5 \times 10) + (9 \times 1)$

Answers to Exercises 2-3d

1. a. $4 \times 4 \times 4$
   b. $3 \times 3 \times 3 \times 3$
   c. 5×5
   d. $2 \times 2 \times 2 \times 2 \times 2$
   e. $2 \times 2 \times 2$
   f. 3×3
   g. $4 \times 4 \times 4 \times 4 \times 4$
   h. $5 \times 5 \times 5 \times 5$

2. a. 12
   b. 64
   c. 81
   d. 10
   e. 25
   f. 32
   g. 6
   h. 8
   i. 9
   j. 20
   k. 20
   l. 1024
   m. 625

3. a. five to the third power
   b. ten to the sixth power
   c. two to the fifth power
   d. ten to the fourth power
   e. two to the third power
   f. eight to the second power

4. a. 4×2
b. 3×10
c. 4×10
d. 5×6
e. 3×8
f. 2
g. 10^3
h. 10^4
i. 6^5
j. 8^3
k. 5×3
l. 3^5
m. 4^2
n. 2×4
o. 5^3
p. 2^3
q. 3×2

5. a. 10^2
b. 10×2
c. 5^4
d. 2^3
e. 2×3
f. 3^{10}
g. 1^{10}
h. 10^1

Answers to Exercises 2-3e

1. a. 3^5
b. 6^6
c. 25^3
d. 5^6
e. 279^5
f. 16^1

2. a. three
b. seven
c. two
d. ten
e. five

3. a. $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$
b. $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$
c. $33 \times 33 \times 33 \times 33 \times 33$
d. $60 \times 60 \times 60 \times 60 \times 60 \times 60$

4. a. 256
b. 36
c. 343
d. 64
e. 81
f. 1000
g. 64
h. 1024

5. a. $2^3 = 8; \quad 3^2 = 9; \quad 3^2 > 2^3$
b. $4^3 = 64; \quad 3^4 = 81; \quad 3^4 > 4^3$

6. a. $(4 \times 10^2) + (6 \times 10) + (8 \times 1)$
b. $(5 \times 10^3) + (3 \times 10^2) + (2 \times 10) + (4 \times 1)$
c. $(7 \times 10^3) + (0 \times 10^2) + (6 \times 10) + (2 \times 1)$
d. $(5 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (2 \times 10) + (6 \times 1)$
e. $(1 \times 10^5) + (0 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (8 \times 10) + (0 \times 1)$

7.

10^1	10	ten
10^2	100	one hundred
10^3	1,000	one thousand
10^4	10,000	ten thousand
10^5	100,000	one hundred thousand
10^6	1,000,000	one million

8. a. 10^3
   b. 10^5
   c. 10^6
   d. 10^8

*9. The exponent of the base "10" tells how many zeros are written to the right of "1" when the numeral is written in the usual way.

10. The value of 10^0 is 1 by definition. The idea should not be stressed here, however. Students can be shown that the meaning of 10^0 is reasonable by the following approach:

$10^4 = 10,000$
$10^3 = 1000$
$10^2 = 100$
$10^1 = 10$
$10^0 = 1$

Each time the exponent is decreased by one, the value of the number is divided by 10. Continuing the process, therefore, it would follow that $10^0 = 1$.

The expanded form of a decimal numeral can be written with powers of ten representing the values of all the places thus: $2156 = (2 \times 10^3) + (1 \times 10^2) + (5 \times 10^1) + (6 \times 10^0)$.

11. 10^{100}. It may be pointed out that 1^{100} is 1. The mathematician in the story is Edward Kasner, and the name "googol" was suggested by his nine-year-old nephew. At the same time, the child suggested that a "googolplex" might be "1" followed by a googol of zeros, or $10^{10^{100}}$. The two terms have caught the public fancy and have become generally accepted in speaking of very large numbers.

2-4. Numerals in Base Five.

The purpose of teaching systems of numeration with bases other than ten is not to produce facility in calculating with such systems. A study of an unfamiliar system aids in understanding a familiar one, just as the study of a foreign language aids us in understanding our own. This understanding will be heightened if the teacher will continually contrast base five with the decimal system. The decimal system is so familiar that its structure and the ideas involved in its algorithms are easily overlooked. In this section attention is focused on numerals, rather than on numbers. Base five was selected for this section rather than base seven or any other base, because it is thought to be easier for pupils.

Questions may arise about the notation for a numeral to base five. We do not write "13_5" because the symbol "5" does not occur in a system of numeration to this base. Replacing the numeral by the written word emphasizes this fact. Note that in any system, the symbol for the base is 10.

It is recommended that this section be developed by a laboratory procedure. Students should be furnished with duplicated sets of counting symbols like those of Exercises 2-4a and 2-4b. Considerable help may be needed in making the transfer from groups of counters to place value numerals.

Devices of any kind in which counters can be manipulated to show successive groups of five will be helpful. For this purpose an open-end abacus can be used, with counters dropped on the rods to indicate various numbers. Pennies can be used as counters, and groups of pennies replaced by nickels and by quarters as numbers become larger. Use such time as is needed and as many approaches as can be devised to develop the concepts of numerals in base five. A clear understanding of this section is necessary before students attempt the remainder of the chapter.

The Celts and Mayans used twenty as a base, probably because they used their toes as well as their fingers in counting. The special name sometimes used for twenty is "score." Some Eskimo tribes probably count by five using the fingers of one hand.

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As a class exercise, pupils enjoy counting orally in base five. Be sure that they say three, four, not thirty-four, for 34_{five}.

Answers to Exercises 2-4a

1. a. 11_{five}
   b. 14_{five}
   c. 22_{five}
   d. 33_{five}
   e. 42_{five}

2. a. $\boxed{x\ x\ x\ x\ x}\ x\ x\ x$
   b. $\boxed{x\ x\ x\ x\ x}\ x$
   $\boxed{x\ x\ x\ x\ x}$
   c. $\boxed{x\ x\ x\ x\ x}$
   $\boxed{x\ x\ x\ x\ x}$
   $\boxed{x\ x\ x\ x\ x}$
   d. $\boxed{x\ x\ x\ x\ x}\ x$
   $\boxed{x\ x\ x\ x\ x}\ x$
   $\boxed{x\ x\ x\ x\ x}\ x$
   $\boxed{x\ x\ x\ x\ x}\ x$

3.

Numeral in Base five	Expanded Form	Numerals in Base Ten
1_{five}	1 × one	1
2_{five}	2 × one	2
3_{five}	3 × one	3
4_{five}	4 × one	4
10_{five}	(1 × five) + (0 × one)	5
11_{five}	(1 × five) + (1 × one)	6
12_{five}	(1 × five) + (2 × one)	7
13_{five}	(1 × five) + (3 × one)	8
14_{five}	(1 × five) + (4 × one)	9
20_{five}	(2 × five) + (0 × one)	10
21_{five}	(2 × five) + (1 × one)	11
22_{five}	(2 × five) + (2 × one)	12
23_{five}	(2 × five) + (3 × one)	13
24_{five}	(2 × five) + (4 × one)	14
30_{five}	(3 × five) + (0 × one)	15
31_{five}	(3 × five) + (1 × one)	16

3. (continued)

Numeral in Base five	Expanded Form	Numerals in Base ten
32_{five}	(3 × five) + (2 × one)	17
33_{five}	(3 × five) + (3 × one)	18
34_{five}	(3 × five) + (4 × one)	19
40_{five}	(4 × five) + (0 × one)	20
41_{five}	(4 × five) + (1 × one)	21
42_{five}	(4 × five) + (2 × one)	22
43_{five}	(4 × five) + (3 × one)	23
44_{five}	(4 × five) + (4 × one)	24

The subscript "five" in the first four numerals in base five is included only for emphasis, since "4" represents the same number, whether the base is five or ten.

Answers to Exercises 2-4b

1. 132_{five}
2. 124_{five}
3. 211_{five}
4. 320_{five}
5. 200_{five}
6. a. twenty-five
   b. 125 or 5 × 5 × 5

7.

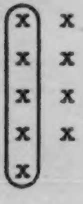
	Quarters	Nickels	Pennies
a.		2	3
b.		4	
c.	1		4
d.	1	4	1
e.	2	1	3
f.	3		
g.	4	3	1
h.	4	4	4

8. a. 23_{five}
   b. 40_{five}
   c. 104_{five}
   d. 141_{five}
   e. 213_{five}
   f. 300_{five}
   g. 431_{five}
   h. 444_{five}

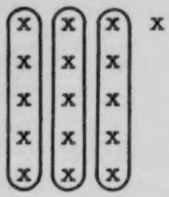
Answers to Exercises 2-4c

1. powers

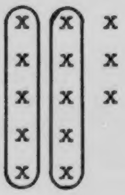
2. a.



b.



c.



3. a. 9
   b. 16
   c. 13

4. a. (1 × five) + (3 × one) = 8
   b. (2 × five) + (4 × one) = 14
   c. (3 × five) + (2 × one) = 17
   d. (4 × five) + (0 × one) = 20
   e. (1 × five^2) + (2 × five) + (3 × one) = 38
   f. (3 × five^2) + (1 × five) + (2 × one) = 82
   g. (2 × five^2) + (2 × five) + (2 × one) = 62
   h. (4 × five^2) + (0 × five) + (3 × one) = 103
   i. (2 × five^3) + (1 × five^2) + (3 × five) + (4 × one) = 294

5. a. 3 twenty-fives, or 3 × five^2 or 3 × five × five.
   b. 3 × five
   c. 3 × one
   d. 3 one hundred twenty-fives, or 3 × five^3 or 3 × five × five × five.

6. There may be many suggestions. Here is an opportunity for ingenuity, though names should suggest meanings. The suggestion of one class of students was 20_{five} = twofi; 30_{five} = thrifi; 40_{five} = forfi; 100_{five} = Fifi (Give it the French pronunciation.)

7. a. 1000_{five} b. 410_{five} c. 4000_{five}

8. 13 years old; 20 guests; 62 hamburgers; 48 doughnuts; 7 quarts of ice cream; 50 bottles of pop; 8 o'clock; 77 cents.

*9. a. 6, 14, 32, 40

b. Last digit divisible by 2 (or even)

c. 4_{five}, 11_{five}, 22_{five}, 123_{five}

d. Sum of the digits is divisible by 2.

If this property of divisibility of base five numerals is not apparent to students, they should be led to investigate more fully. Awareness of the property will not only be enjoyable to the student, but will aid him later in discovery of the test for divisibility by 3 in decimal numerals.

*10. a. 20, 30, 50

b. Last digit is zero.

c. 20_{five}, 30_{five}, 40_{five}

d. Last digit is zero.

e. A digit in the one place shows the number of objects not included in any group, no matter what base is used. In base ten the final zero shows no remainder when grouping by ten is done; in base five the final zero shows no remainder when grouping by five is done.

2-5. Addition and Subtraction in Base Five.

Computation with base five will probably be more difficult for the students than counting or writing numerals. The explanatory paragraphs in all the computation sections should be presented by the teacher or read through with the class to be sure that all the steps are clear. Exercises should be assigned as homework only after enough class discussion has clarified methods of procedure for the pupils.

Addition in base five is undertaken to clarify addition in decimal notation. Some of the newer elementary school textbooks prefer to use the word "change" or "regroup" rather than "borrow" or "carry" since the first two words seem to describe the actual process better than the last.

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Point out to the pupils that in working in base ten it is often necessary to regroup ten ones as one ten, whereas in base five we regroup five ones as one five.

As pupils use the addition table for subtraction, they will observe that subtraction is the inverse of addition.

In computing with base five, pupils may find writing the subscript "five" irritating because it consumes so much time. It has been written in the student text for emphasis, but any agreement made by the class as a means of indicating the base should be satisfactory.

Answers to Exercises 2-5a

1. Be sure that pupils understand the construction of the addition table for base ten:

Addition, Base Ten

+	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10	11
2	2	3	4	5	6	7	8	9	10	11	12
3	3	4	5	6	7	8	9	10	11	12	13
4	4	5	6	7	8	9	10	11	12	13	14
5	5	6	7	8	9	10	11	12	13	14	15
6	6	7	8	9	10	11	12	13	14	15	16
7	7	8	9	10	11	12	13	14	15	16	17
8	8	9	10	11	12	13	14	15	16	17	18
9	9	10	11	12	13	14	15	16	17	18	19
10	10	11	12	13	14	15	16	17	18	19	20

If pupils know the facts, no time should be wasted on the table after its characteristics have been discussed.

2. Pupils should be helped to observe the symmetry of the table with respect to the diagonal. They will notice that $8 + 6 = 6 + 8$, for example, and that this is true for any pair of numbers. Later they will learn that this is the commutative property of addition. The word "commutative" should not be used at this time.

3.

Addition, Base Five

+	0	1	2	3	4	10
0	0	1	2	3	4	10
1	1	2	3	4	10	11
2	2	3	4	10	11	12
3	3	4	10	11	12	13
4	4	10	11	12	13	14
10	10	11	12	13	14	20

There is no value in memorizing this table. The process is more important than the facts. The point to be emphasized is that numbers and number properties are independent of the numerals or symbols used to represent the numbers. Commutativity holds in base five as well as base ten because it is a property of numbers, not numerals.

The table should be kept in the pupil's notebook, or a wall-chart may be made for reference when subtraction exercises are done.

4. Discussion should include observations such as those mentioned for the base ten table.

5. In each case (10 and 10_{five}) the "1" shows one of the base collection. The 13 therefore is one ten plus three, while 13_{five} is one five plus three.

6. 20_{five} = ten and 20 = twenty, but in each case the "2" shows the result of adding two of the smallest groups.

Answers to Exercises 2-5b

1. $34_{five} = 19$
2. $42_{five} = 22$
3. $43_{five} = 23$
4. $334_{five} = 94$
5. $404_{five} = 104$
6. $400_{five} = 100$
7. $130_{five} = 40$
8. $231_{five} = 66$

9. a. 4_{five}
   b. 3_{five}
   c. 2_{five}
   d. 3_{five}

10. a. 4_{five}
    b. 4_{five}
    c. 2_{five}

Answers to Exercises 2-5c

1. a. 3_{five}
   b. $24_{five} = 14$
   c. $122_{five} = 37$
   d. $110_{five} = 30$
   e. $22_{five} = 12$
   f. $121_{five} = 36$

2. Add 27 and 36. The result should give the minuend, 63.

3. Answers will be minuends of 1(a), (b), and (c).

2-6. Multiplication in Base Five.

The extent to which this section and the one succeeding it are used will vary with the class. For good students, the two harder processes will be challenging, and will give an opportunity for discussion of reasons underlying the algorithms of multiplication and division. On the other hand, if addition and subtraction have been very difficult to motivate for a group of children, it may be better to omit these sections or to use them only for demonstration. Ability to compute with base five numerals has no value in itself.

Answers to Exercises 2-6a

1. a.

Multiplication, Base Ten

×	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

b. (1) The product of 0 and any number is zero.

(2) The product of 1 and any number is the number itself.

(3) The order in multiplication does not affect the product.

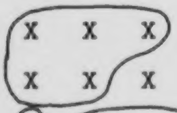
(4) The products marked by the diagonal line are second powers of the counting numbers.

(5) The successive products in any one row or column may be found by adding the same number one more time.

c. Yes.

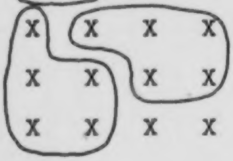
d. 4; yes.

2. a.



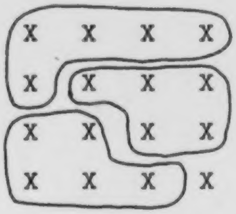
$= 11_{five}$

b.



$= 22_{five}$

c.



$= 31_{five}$

3. a.

Multiplication, Base Five

×	0	1	2	3	4	10
0	0	0	0	0	0	0
1	0	1	2	3	4	10
2	0	2	4	11	13	20
3	0	3	11	14	22	30
4	0	4	13	22	31	40
10	0	10	20	30	40	100

Study of this table is valuable for the additional insight it affords into the understanding of multiplication. There is no value in memorizing it. The table may be used to emphasize that division is the inverse of multiplication.

b. Discussion as for Problem 1(b).

c. Yes

d. 112, 201

4. Base five is easier, because there are fewer products to be learned.

*Answers to Exercises 2-6b

1. 44_{five} = 24
2. 132_{five} = 42
3. 141_{five} = 46
4. 212_{five} = 57
5. 432_{five} = 117

6. 1322_{five} = 212
7. 2021_{five} = 261

*8. 2144_{five} = 299

*9. 30313_{five} = 1958

---

*2-7. Division in Base Five.

Since division is the most demanding operation, it is suggested that teachers regard the topic as optional and do only as much as they judge appropriate, in class discussion. Pupils may need help in learning how to use the multiplication pages 53-54:

table to find division facts. Exercises are included for those pupils who wish to attempt them.

*Answers to Exercises 2-7a

1. a. 2 c. 4 e. 2 R 2
   b. 4 d. 4 R 2 f. 4 R 3

2. a. 41_{five}
   b. 24_{five}
   c. $\text{242 R 2}_{\text{five}}$
   d. 23_{five}
   e. $\text{33 R 12}_{\text{five}}$

Answers to Review Exercises 2-7b

1. a. $(3 \times \text{five}^2) + (0 \times \text{five}) + (2 \times \text{one})$
   b. $(1 \times \text{ten}^2) + (6 \times \text{ten}) + (7 \times \text{one})$

2. $302_{\text{five}} = 77$ 167 is larger.

3. a. 111_{five}
   b. 340_{five}

4. a. 23_{five}
   b. 213_{five}

5. a. 202_{five}
   b. 234_{five}

6. Room 123; book 7; 15 chapters; 394 pages; 32 pupils; 5 times; 55 minutes; 13 girls; 19 boys; 11 years old; 66 inches tall.

7. a. 37; 136; 87; 59; 28; 3278; 13; 9.
   b. 1041_{five}
   c. 146
   d. 42_{five}; 22

2-8. Changing Decimal Numerals to Base Five Numerals.

Ask for the highest power of five which is contained in the number given in base ten numeration. For example, consider 283. Is five^4 (or 625) contained in 283? Is five^3 (or 125)? After we have taken as many 125's as possible from 283, how much remains? The next power of five is five^2. How many 25's are contained in 33? Finally, how many 5's and how many 1's are left?

Answers to Exercises 2-8

1. a. $17 = (3 \times 5) + 2 = 32_{five}$
   b. $36 = (1 \times 25) + (2 \times 5) + 1 = 121_{five}$
   c. $68 = (2 \times 25) + (3 \times 5) + 3 = 233_{five}$
   d. $75 = (3 \times 25) + (0 \times 5) + 0 = 300_{five}$
   e. $92 = (3 \times 25) + (3 \times 5) + 2 = 332_{five}$
   *f. $183 = (1 \times 125) + (2 \times 25) + (1 \times 5) + 3 = 1213_{five}$

2. a. 24_{five}
   b. 43_{five}
   c. 122_{five}
   d. 211_{five}
   e. 311_{five}
   *f. 1002_{five}
   *g. 2022_{five}

2-9. Numerals in Other Bases.

Bring out the idea that the base of the system that we use is "ten" for historical rather than mathematical reasons. Some mathematicians have suggested that a prime number such as 7 has certain advantages. The Duodecimal Society of America, 20 Carlton Place, Staten Island 4, New York supports the adoption of twelve as the best number base. Information about the duodecimal system is furnished by this society on request. Exercises in other number bases help establish an understanding of what a positional, power system of numeration is.

The Binary and Duodecimal Systems

The use of binary notation in high speed computers is well known. The binary system is used for computers since there are only two digits, and an electric mechanism is either "On" or

"Off." Such an arrangement is called a flip-flop mechanism. A number of pamphlets distributed by IBM, Remington Rand, and similar sources may be obtained by request and used for supplementary reading and study. "*Yes No - One Zero*" published by Esso Standard Oil Co., 15 West 51st., New York 19, New York is available for the asking only in states served by Esso.

For a discussion of a binary computer see Teachers' Commentary Vol. 1, Part 1, SMSG *Mathematics for Junior High School*, pp. 36-40.

It should be of interest that the sum 11001 + 110 looks the same in the binary system, decimal system, and, in fact, all positional numeral systems. The meaning, however, is quite different.

The base two has the disadvantage that, while only two different digits are used, many more digits are needed to express numbers in binary notation than in decimal, e.g.,

$$2000_{ten} = 11{,}111{,}010{,}000_{two}$$

Here is a set of cards which can be used in a number trick.

1			
1	9	17	25
3	11	19	27
5	13	21	29
7	15	23	31

2			
2	10	18	26
3	11	19	27
6	14	22	30
7	15	23	31

4			
4	12	20	28
5	13	21	29
6	14	22	30
7	15	23	31

8			
8	12	24	28
9	13	25	29
10	14	26	30
11	15	27	31

16			
16	20	24	28
17	21	25	29
18	22	26	30
19	23	27	31

Using the first four cards, tell a person to choose a number between 1 and 15, to pick out the cards containing that number and to give them to you. By adding the numbers at the top of the cards he gives you, you can tell him the number he chose. Note that the numerals at the top of the cards represent the powers of two in reverse order.

By using all five cards, you can pick out numbers from 1 to 31. The trick is based on the application of the binary numbers. Further information may be found in the following volumes:

Jones, Philip S. *Understanding Numbers: Their History and Use*
Merrill, Helen A. *Mathematical Excursions*
Swain, Robert L. *Understanding Arithmetic*

If you have a peg board and some match sticks, you can represent base two numbers on the board. Leave a hole blank for 0 and put in a match stick for one. Represent two numbers on the board, one below the other, and try adding on the board.

The twelve system uses two digits more than the decimal system. From some points of view twelve is a better choice for a base than ten. Many products are packaged and sold by the dozen and by the gross. Twelve is divisible by 2, 3, 4, and 6 as well as 12. Ten is divisible only by 2, 5, and 10. Because it employs a larger base, large numbers may be represented in base twelve with fewer digits than smaller bases require. For example:

$$\mathrm{TOE}_{\text{twelve}} = 1451_{\text{ten}}$$

Answers to Exercises 2-9a

1. a. The numeral *says* there are two groups of seven and six more.
   b. two, six, base seven
   c. twenty

2. 22_{three}
3. 34_{five}
4. 16_{twelve}
5. 50_{seven}
6. 111_{two}
7. 212_{three}
8. 10_{five}; 5_{seven}; 11_{four}
9. 13_{five}; 11_{seven}; 20_{four}

10. 21_{five}; 14_{seven}; 23_{four}

11. 30_{five}; 21_{seven}; 33_{four}

12. 44_{five}; 33_{seven}; 120_{four}

Answers to Exercises 2-9b

1. 35_{seven}

2. a. (X X X X X X X) X

   b. (X X X X X X X) (X X X X X X X) X X X X X

3.

Base ten	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Base seven	1	2	3	4	5	6	10	11	12	13	14	15	16	20

ten	15	16	17	18	19	20	21	22	23	24	25
seven	21	22	23	24	25	26	30	31	32	33	34

4. 49

5. 15_{seven}

6. 6

7. a. 55_{seven} b. 126_{seven} c. 44_{seven}

8. one, seven, forty-nine
   three hundred forty-three

Answers to Exercises 2-9c

1.

Base ten	15	16	17	18	19	20	21	22	23	24	25	26
Base twelve	13	14	15	16	17	18	19	1T	1E	20	21	22
ten	27	28	29	30	31	32	33	34	35	36	37	38
twelve	23	24	25	26	27	28	29	2T	2E	30	31	32
ten	39	40	41	42	43	44	45	46	47	48	49	50
twelve	33	34	35	36	37	38	39	3T	3E	40	41	42

2. One hundred forty-four

3. 84_{twelve}

4. 83; 125; 131; 58

5. base twelve

6. one, twelve, twelve^2 (or 144)

Answers to Exercises 2-9d

1. a. 12 b. 5 c. 7 d. 10 e. 2

2. 2

3. a. two × two = four
   b. five × five = twenty-five
   c. twelve × twelve = one hundred forty-four.

4. a. Addition, Base Two

+	0	1
0	0	1
1	1	10

   b. There are only four addition facts.

5. a. Multiplication, Base Two

×	0	1
0	0	0
1	0	1

   b. There are only four multiplication facts.
   c. The two tables are not alike, except that 0 + 0 and 0 × 0 both equal 0.
   d. The binary system is very simple because there are only four addition and four multiplication facts to remember. Computation is simple.
   e. Numerals for large numbers are too long.

6. a. 111_{two}
   b. 1100_{two}

7.

Ten	Two	Five	Eight
1	1	1	1
2	10	2	2
5	101	10	5
7	111	12	7
15	1111	30	17
16	10,000	31	20
32	100,000	112	40
64	1,000,000	224	100
256	100,000,000	2011	400

*8. a. 24_{ten}; 33_{seven}
   b. 32_{five}; 15_{twelve}
   c. 62_{ten}; 332_{four}
   d. 11011_{two}; 1000_{three}

9. In the octal system, each digit corresponds to a group of three places in the binary system.

$$\overbrace{111}^{7},\ \overbrace{010}^{2},\ \overbrace{110_{two}}^{6_{eight}}$$

10. a. weights; 1 oz., 2 oz., 4 oz., 8 oz.
   b. five weights, those listed in "a" and 16 oz.

2-11. Chapter Review.

Answers to Exercises 2-11

1. a. Two thousand, thirty-five
   b. Fifty-six thousand, two hundred eight
   c. Eight hundred seventy-six million, five hundred thousand, two hundred ten

2. a. 32 b. 251 c. 19 d. 900

3. 8

4. a. $(2 \times 10^3) + (3 \times 10^2) + (1 \times 10) + (4 \times 1)$
   b. $(1 \times \text{five}^3) + (3 \times \text{five}^2) + (0 \times \text{five}) + (4 \times 1)$
   c. $(1 \times \text{two}^2) + (1 \times \text{two}) + (1 \times 1)$
   d. $(1 \times \text{seven}^2) + (2 \times \text{seven}) + (6 \times 1)$

5. Ten

6. All true except (d).

7. a. 625 c. 49 e. 64 g. 32
   b. 16 d. 27 f. 10

8. Base 20; 87 years.

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9. a. three
b. six
c. four
d. six
e. nine

10. a. twelve b. seven c. five d. two

11. Since there are only five symbols, we assume that this is a base five system. Therefore DCBAO = $(4 \times 625) + (3 \times 125) + (2 \times 25) + (1 \times 5) + 0 = 2930$. A rare student may point out that any system with base of five or more might use these symbols for the numbers from zero to four. In such cases, we would not know the value of DCBAO without further information.

12.

1	2	3	4	5	6	7	8	9	10
A	B	C	D	AO	AA	AB	AC	AD	BO
11	12	13	14	15	16	17	18	19	20
BA	BB	BC	BD	CO	CA	CB	CC	CD	DO

Sample Test Questions for Chapter 2

Part I. True - False

1. The 3 in 356_{seven} stands for three hundred.
2. 10^4 means $10 \times 10 \times 10 \times 10$.
3. The numeral 8 means the same number in the ten system and in the twelve system.
4. The smaller the base, the more basic combinations there are in the multiplication table.
5. The fourth place from the right in the decimal system has the place value 10^5.
6. In base two numerals the number after 100 is 1000.
7. We can make a symbol to mean what we wish.
8. When we "carry" in addition the value of what is carried depends upon the base.
9. A number may be written in numerals with any whole number greater than one as a base.
10. In the symbol 6^3, the exponent is 3.
11. 513_{six} means $(5 \times six \times six \times six) + (1 \times six \times six) + (3 \times six)$.
12. The 1 in 10,000 (base two) means 1×2^4 or sixteen.
13. The following numerals represent the same number: 183_{twelve}: 363_{eight}; 1033_{six}.
14. In base eight numerals, the number before 70 is 66.
15. Four symbols are sufficient for a numeration system with base five.
16. In the base four system $3 + 3 = 11_{four}$
17. When we "borrow" in the twelve system as in 157 - 6E, we actually "borrow" twelve.
18. In the Egyptian system a single symbol could be used to represent a collection of several things.
19. The Roman numeral system had a symbol for zero.

Part II. Completion

1. In decimal numerals 14_{twelve} is ______.
2. MCXXIV in decimal numerals is ______.
3. The decimal system uses ______ different symbols.
4. In any numeration system, the smallest place value is ______.
5. 629,468,000 written in words is ______________________________

______________________.
6. The number represented by 212_{seven} is ______(even, odd)
7. In expanded notation $5{,}678_{\text{ten}}$ is ____________.
8. $213_{\text{five}} + 312_{\text{five}} =$ ____________$_{\text{five}}$.
9. Multiply: $32_{\text{four}} \times 3_{\text{four}} =$ ______.
10. $110011_{\text{two}} =$ __________$_{\text{ten}}$.
11. The numeral 444_{five} represents an ________ (even, odd)
12. Add: $62_{\text{seven}} + 16_{\text{seven}} =$ _______.
13. $13_{\text{ten}} =$ _________$_{\text{two}}$.
14. The numeral after 37_{eight} is _______eight.
15. The largest possible number that can be represented by the digits 5, 6, 7, and 0 is _______.
16. The smallest possible number that can be represented by the digits 5, 6, 7, and 0 is _______.
17. The largest number that can be represented without exponents, using only two 4's is _______.
18. Write this numeral without exponents: 5^3 _______.
19. The numeral immediately before 1000_{two} is _______.
20. Subtract: $42_{\text{five}} - 14_{\text{five}} =$ _______.

Part III. Multiple-Choice

I. In which of the numerals below does 1 stand for four?

a. 21_{four}

b. 21_{eight}

c. 100_{two}

d. 102_{three}

e. None of the above is correct.

2. In what base are the numerals written if $2 \times 2 = 10$?

a. Base two

b. Base three

c. Base four

d. Base five

e. All of the above are correct.

3. A decimal numeral which represents an odd number is:

a. 461,000

b. 7629

c. 5634

d. 9,000,000

e. None of the above is correct

4. If N represents an even number, the next consecutive even number can be represented by:

a. N

b. $N + 1$

c. $N + 2$

d. $2N$

e. All of the above are correct.

5. Which numeral represents the largest number?

a. 43_{five}

b. 212_{three}

c. 10110_{two}

d. 24_{nine}

e. $10_{twenty\text{-}five}$

6. Which is correct?

a. $5^4 = 5 + 5 + 5 + 5$

b. $4^3 = 4 \times 4 \times 4$

c. $5^4 = 4 \times 4 \times 4 \times 4 \times 4$

d. $2^3 = 2 \times 3$

e. None of the above is correct

7. 6120_{nine} is how many times as large as 612_{nine}?

a. twelve
b. ten
c. nine
d. five
e. None of the above is correct.

8. In which base does the numeral 53 represent an even number?

a. twelve
b. ten
c. eight
d. seven
e. six

Answers to Sample Test Questions for Chapter 2

Part I True-False	Part II Completion	Part III Multiple Choice
1. False	1. 16	1. c
2. True	2. 112^4	2. c
3. True	3. Ten	3. b
4. False	4. One	4. c
5. False	5. Six hundred twenty-nine million, four hundred sixty-eight thousand	5. e
6. False	6. Odd	6. b
7. True	7. $(5 \times 10^3) + (6 \times 10^2) + (7 \times 10^1) + (8 \times 1)$	7. c
8. True	8. 1030_{five}	8. d
9. True	9. 222_{four}	
10. True	10. 51	
11. False	11. Even	
12. True	12. 111_{seven}	
13. False	13. 1101_{two}	
14. False	14. 40_{eight}	
15. False	15. 7650	
16. False	16. 5067	
17. True	17. 44	
18. True	18. 125	
19. False	19. 111_{two}	
	20. 23_{five}	

CHAPTER 3

WHOLE NUMBERS

3-1. Introduction.

This chapter is designed to help the youngster grasp the concept of counting and the properties which govern the fundamental operations with the counting numbers and the whole numbers. The new vocabulary has been introduced so that student and teacher may communicate more efficiently and effectively. Continual classroom usage of such words as commutative, associative, distributive, and inverse should help to make these an integral part of the student's mathematical vocabulary. There are a large number of exercises so that the student will have an opportunity to practice these new concepts and also maintain a satisfactory level of achievement with the manipulative skills. Small numbers have purposely been used in many of the exercises so that complex arithmetic operations will not interfere with the student's understanding of the properties with which he will be working.

It is estimated that 22-25 days will be required to complete this chapter.

3-2. Sets.

Emphasis here is placed upon the meaning of set. The concept of set has been introduced to facilitate the definition of counting numbers. This same concept will also enable us to define closure more adequately, and to discuss the properties of the counting numbers and of the whole numbers. It is important that the student comprehend this concept for later use with nonmetric geometry, prime numbers, and all of his work this year. The class will enjoy talking about such sets as:

a. the set of brown-eyed boys in the room.
b. the set of blue-eyed girls in the room.
c. the set of girls over 5 feet in height who are in the room.

The teacher may use numerous illustrations to indicate that a set may have any number of elements.

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There has been no attempt to discuss the union and intersection of two or more sets. Intersection will be developed in Chapter 4, NON-METRIC GEOMETRY I, and union in Chapter 7, NON-METRIC GEOMETRY II. It is suggested that the teacher avoid these concepts at this time, since our objective is merely to introduce the meaning of a set.

Answers to Discussion Exercises 3-2a

1. There are many such words with which the student is already familiar. Some of these might be:
   a. pack of matches
   b. baseball team
   c. my gang
   d. flock of sheep
   e. swarm of bees
   f. family of people
   g. pair of cuff links
   h. herd of cows
2. Set of chairs, set of desks, set of windows, set of books, set of boys, set of girls, are just a few of the many examples which might be mentioned.
3. Set of dishes, set of furniture, set of silverware, set of spoons, set of closets, are just a few.

Answers to Exercises 3-2b

1. M = {April, August}
2. D = {Sunday, Saturday}
3. There are many possible answers which are correct.
4. S = {Maine, Maryland, Minnesota, Missouri, Mississippi, Montana, Michigan, Massachusetts}
5. There are many correct solutions depending upon your particular school.
6. R = {10, 12, 14, 16, 18, 20, 22, 24}
7. A = {w, x, y, z}
8. B = {23, 25, 27, 29, 31, 33}

In the following exercises it should be noted that each set may be described correctly in more than one way. Only one possible description is given below.

9. The set of capitals of the United States, England, and France.
10. The set of odd numbers greater than 11 but smaller than 19.
11. The set of current American coins less than one dollar in value.
12. The set of the first 6 letters of the alphabet.
13. The set of numbers from 3 to 21 inclusive, which are exactly divisible by 3.
14. The set of states whose names begin with A.
15. The set of one-digit numerals.
16. The set of all odd numbers from 1 to 9 inclusive.

3-3. Counting Numbers.

The teacher should strive to develop understanding of the following concepts:

1. The number of members of any set can be found by matching the members of the set with the members of some standard set. This is a clumsy method if the number of members is large since the standard sets must themselves be large. The best known way of finding the number of members of a set, then, is by matching the members of the set with a memorized set of sounds representing the counting numbers.

2. The counting numbers are represented by the set $\{1, 2, 3, 4, \ldots\}$ and do not include zero. The counting numbers are often called natural numbers and the teacher may wish to point this out to the student. We have chosen to use the name "counting number" since it is already familiar to many members of the class.

3. The counting numbers and zero constitute the set called the whole numbers. It is necessary that the students be fully aware of the difference between the set of counting numbers and the set of whole numbers to avoid difficulties later when working with the properties of operations.

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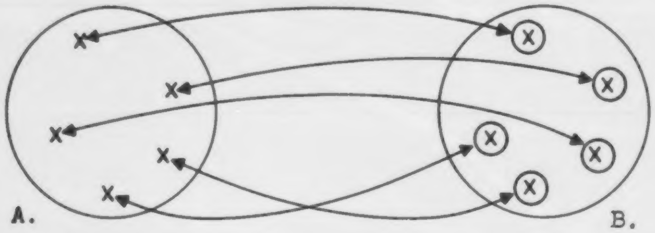
The idea that we want to get across here is that by counting we have a set of numbers that "matches" the objects. The one-to-one correspondence is a pairing of the things we are counting with a subset of the counting numbers. The set of all counting numbers never ends, but the counting of objects does. When we have two finite sets of objects that have the same number of elements, we can pair them so that each element of set A corresponds to exactly one element of set B *and* each element of set B corresponds back to that element of set A.



Answers to Exercises 3-3

1. a. {North America, South America, Africa, Europe, Asia, Australia, Antarctica}
   b. {Atlantic, Pacific, Indian, Antarctic, Arctic}
   c. {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}
   d. {1, 2, 3, ..., 10}
   e. {0, 1, 2, 3, 4}
   f. {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}
   h. {1, 4, 9, 16, 25}
2. Yes. 8 was left out.
3. a. 0 and 11
   b. 0, 2, 5, and 11
4. 17
5. The following shows a one-to-one correspondence between the *counting* numbers and the *even* numbers.
6. He subtracted 27 from 81. The correct answer is (54 + 1).

3-4. Properties of Operations.

The principal objectives in the study of the commutative, associative, and distributive properties are to have the pupils understand the statements in mathematical language; to distinguish one property from another; and to recognize the one, or ones, that may be used in various exercises. These are not properties that are being proved. The pupils have used them for a long time, but they probably have not had names for them and have not recognized when they have been using them.

3-5. Commutative Property.

The exercises are designed to help the student discover that both addition and multiplication are commutative operations, but that subtraction is not commutative nor is division a commutative operation. To help cement the commutative property of multiplication, it might be helpful to arrange stars on a cardboard in the following manner:

* * * * *
* * * * *
* * * * *
* * * * *

This arrangement shows 4 rows of 5 stars in a row. However, by rotating the cardboard 90°, the arrangement will show 5 rows of 4 stars in a row.

The meaning of such new symbols as $<$, $>$, $\nless$, $\cdot$, are to be discovered by the student. However, an opportunity has been provided for the teacher to insure that each member of the class is in agreement on what these symbols represent.

Since it is impossible to list all pairs of numbers in addition and multiplication, letters have been introduced to generalize the commutative properties for addition and multiplication. The large variety of exercises should lead naturally into this generalization provided the student understands that letters may represent any number whatsoever.

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Other examples of commutative activities might be:

To wash your face and wash your hair.

To go north one block and then west one block.

To count to 100 and write the alphabet.

Other examples of activities which are not commutative:

To put out the cat and go to bed.

To write a word and erase that word.

Answers to Class Exercises 3-5a

1. a. 3
   b. 24
   c. 82
   d. 3
   e. 6
   f. 2
   g. 9
   h. 6
   *i. a
   *j. b
   *k. d
   *l. c

2. The results in parts a, b, c, and d remain the same. The others change because the commutative property does not hold for division and subtraction.

3. a. No. Addition is commutative.
   b. No. Multiplication is commutative.
   c. Yes. Subtraction is not commutative.
   d. Yes. Division is not commutative.

Answers to Exercises 3-5b

1. Addition and multiplication are commutative. Subtraction and division are not.

2. The activities are commutative in parts a and d.

3. a. 63 b. 57 c. 79 d. 1051 e. 1481

4. a. 782 b. 800 c. 5073 d. 183,314 e. 543,648

Answers to Exercises 3-5c

1. 2 is less than 6 or 2 is smaller than 6.
2. 3 times 7 equals 21.
3. 3 is not equal to 2.
4. 8 is not equal to 11.

5. 14 plus 15 is less than 16 plus 18.
6. 8 times 25 equals 200.
7. 92 is greater than 25.
8. 9 times 8 equals 72.
9. 4 is not equal to 17.
10. 11 is greater than 6.
11. 19 minus 17 is less than 5.
12. 14 divided by 7 is less than 5.
13. 16 is greater than 8 and 8 is greater than 2.
    Or, 8 is between 16 and 2.
14. 3 is less than 10 and 10 is less than 14.
    Or, 10 is between 3 and 14.

Answers to Exercises 3-5d

1. =
2. >
3. >
4. =
5. <
6. <
7. =
8. <
9. =
10. =
11. >, >
12. <, <
13. >
14. =
15. >
16. <

Answers to Exercises 3-5e

1. {6}
2. {5}
3. {1}
4. {0, 1}
5. {0, 1, 2, 3, 4}
6. {4}
7. {0, 1, 2, 3, 4, 5}
8. {0, 1, 2, 3, 4}
9. set of whole numbers
10. {0, 1}
11. {9}
12. {8}
13. {0, 1, 2, 3, 4, 5, 6, 7, 8}
14. Set of all whole numbers.

3-6. The Associative Property.

Have the students use blocks or disks to make such arrangements as

* * *   * *   * * * *

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Have them push the first two sets together and count the total $(3 + 2) + 4$. After rearranging, have them push the second two sets together and count the total $3 + (2 + 4)$. Use sufficient variations of this procedure to lead to the understanding that $(a + b) + c = a + (b + c)$ where a, b, and c are any whole numbers.

Then ask: Is the product $(3 \cdot 4) \cdot 5$ equal to the product $3 \cdot (4 \cdot 5)$?

This may be illustrated by arranging a set of 20 blocks in a rectangular array, 4 by 5. Then put two layers of 20 blocks each on top of these forming a box arrangement. Look at it in different ways to see $(3 \cdot 4) \cdot 5$ and $3 \cdot (4 \cdot 5)$. Different boxes may be made to illustrate $2 \cdot (3 \cdot 4)$, $(2 \cdot 3) \cdot 4$ and many others. Again, emphasis is upon arrival at understanding that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ where a, b, and c are any whole numbers.

Sufficient exercises have been provided so that the student will soon realize that there is an associative property for addition and multiplication, but not for subtraction and division.

It is suggested that part 5 of Exercises 3-6c be done in class to assure maximum understanding.

Point out some operations or activities which are not associative and have students suggest others.

Answers to Oral Exercises 3-6a

1. $11 + 2 = 4 + 9$
   $13 = 13$
2. $119 + 98 = 46 + 171$
   $217 = 217$
3. $34 + 16 = 41 + 9$
   $50 = 50$
4. $21 + 9 = 26 + 4$
   $30 = 30$
5. The associative property is used in these examples.

Answers to Exercises 3-6b

1. a. $$\begin{aligned}(10 + 5) + 3 &= 10 + (5 + 3)\\ &= 10 + 8\\ &= 18\end{aligned}$$

b. $$\begin{aligned}(30 + 3) + 6 &= 30 + (3 + 6)\\ &= 30 + 9\\ &= 39\end{aligned}$$

c. $$\begin{aligned}(70 + 2) + 5 &= 70 + (2 + 5)\\ &= 70 + 7\\ &= 77\end{aligned}$$

d. $$\begin{aligned}(90 + 6) + 7 &= 90 + (6 + 7)\\ &= 90 + 13\\ &= 103\end{aligned}$$

e. $$\begin{aligned}(30 + 4) + 2 &= 30 + (4 + 2)\\ &= 30 + 6\\ &= 36\end{aligned}$$

2. a. $(51 + 9) + 22 = 82$
b. $16 + (25 + 25) = 66$
c. $(311 + 89) + 76 = 476$
d. $15 + (14 + 16) = 45$
e. $(23 + 17) + 18 = 58$
f. $(24 + 6) + 87 = 117$

Answers to Exercises 3-6c

1. a. $$\begin{aligned}7 \cdot 12 &= 21 \cdot 4\\ 84 &= 84\end{aligned}$$

b. $$\begin{aligned}45 \cdot 2 &= 5 \cdot 18\\ 90 &= 90\end{aligned}$$

c. $$\begin{aligned}21 \cdot 15 &= 63 \cdot 5\\ 315 &= 315\end{aligned}$$

d. $$\begin{aligned}9 \cdot 16 &= 18 \cdot 8\\ 144 &= 144\end{aligned}$$

2. a. 74 b. 42 c. 79 d. 6

3. a. No b. No c. The associative property does not hold for subtraction.

4. a. No
   b. No
   c. $(75 + 15) + 5$
   d. $75 + (15 + 5)$
   e. $80 + (20 + 2)$
   f. $(80 + 20) + 2$
   g. The associative property does not hold for division.

5. It is suggested that these exercises be done orally if at all possible.
   a. 16
   b. 260
   c. 1080
   d. 22
   e. 7600
   f. 922
   g. 670
   h. 216

3-7. The Distributive Property.

Emphasize that the distributive property is the connecting link between the two operations of addition and multiplication. However, multiplication is distributive over addition but addition is not distributive over multiplication. This says that $a \cdot (b + c) = a \cdot b + a \cdot c$. We cannot do anything to simplify $a + (b \cdot c)$; that is, $a + (b \cdot c)$ is not equal to $(a + b) \cdot (a + c)$.

Blocks can be used in the following way. Lay out 2 rows of 3 each and 2 rows of 5 each.

```
* * *      * * * * *
* * *      * * * * *
```

Ask: If we move these together, we will have 2 times what number? When they move them together do they get 2 times 8? This can be repeated until they understand that

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c),$$

when a, b, and c are whole numbers. Repetition of the same illustration with different numbers of blocks may be better than different types of illustrations.

It is very important that the student grasp the idea that the distributive property involves two operations; namely, addition and multiplication. However, it is equally important that the student realize that not all problems involving both multiplication and addition utilize this property. This fact is brought out in Problem 1 of Exercises 3-7b.

Use the distributive property to help make mental computations during class, such as:

$$7 \cdot 32 = 7 \cdot (30 + 2) = (7 \cdot 30) + (7 \cdot 2) = 210 + 14 = 224$$

$$35 \cdot 8 = (30 + 5) = 8 \cdot (30 \cdot 8) + (5 \cdot 8) = 240 + 40 = 280$$

Answers to Exercises 3-7a

1. a. 45
   b. 135
   c. 45
   d. 63
   e. 60
   f. 60
   g. 30
   h. 30
   i. 45
   j. 72

2. a. $4 \cdot 12 = 28 + 20$
   $48 = 48$

   b. $18 + 24 = 6 \cdot 7$
   $42 = 42$

   c. $48 + 42 = 15 \cdot 6$
   $90 = 90$

   d. $23 \cdot 5 = 46 + 69$
   $115 = 115$

   e. $11 \cdot 7 = 33 + 44$
   $77 = 77$

   f. $30 + 18 = 6 \cdot 8$
   $48 = 48$

   g. $2 \cdot 20 = 24 + 16$
   $40 = 40$

   h. $48 + 16 = 16 \cdot 4$
   $64 = 64$

   i. $12 + 24 = 3 \cdot 12$
   $36 = 36$

3. a. $3 \cdot (4 + \underline{3}) = (3 \cdot 4) + (3 \cdot 3)$
   b. $2 \cdot (\underline{4} + 5) = (2 \cdot 4) + (\underline{2} \cdot 5)$, There are other possible answers, but only this pair demonstrates the distributive property.
   c. $13 \cdot (6 + 4) = (13 \cdot \underline{6}) + (13 \cdot \underline{4})$
   d. $(2 \cdot 7) + (3 \cdot \underline{7}) = (\underline{2} + \underline{3}) \cdot 7$
   e. $(\underline{6} \cdot 4) + (\underline{7} \cdot 4) = (6 + 7) \cdot 4$

4. a. $4 \cdot 2 + 4 \cdot 3$
   b. $7 \cdot 4 + 7 \cdot 6$
   c. $9 \cdot (8 + 2)$
   d. $6 \cdot 13 + 6 \cdot 27$
   e. $12 \cdot (5 + 7)$
   f. $5 \cdot (6 + 7)$
   g. $8 \cdot 14 + 8 \cdot 17$
   h. $6 \cdot 5 + 13 \cdot 5$
   i. $(5 + 4) \cdot 12$
   j. $3 \cdot 4 + 5 \cdot 4$

5. a. $(2 \cdot 3) + (2 \cdot 2)$ or $2 \cdot (3 + 2)$
   b. $(3 \cdot 4) + (3 \cdot 3)$ or $3 \cdot (4 + 3)$
   c. $(5 \cdot 2) + (5 \cdot 3)$ or $5 \cdot (2 + 3)$
   d. $(3 \cdot 1) + (3 \cdot 2)$ or $3 \cdot (1 + 2)$
   e. $(3 \cdot 4) + (3 \cdot 5)$ or $3 \cdot (4 + 5)$
   f. $(5 \cdot 3) + (5 \cdot 5)$ or $5 \cdot (3 + 5)$
   g. $(5 \cdot 7) + (5 \cdot 8)$ or $5 \cdot (7 + 8)$
   h. $(3 \cdot 10) + (3 \cdot 7)$ or $3 \cdot (10 + 7)$
   i. $(3 \cdot 9) + (3 \cdot 17)$ or $3 \cdot (9 + 17)$
   j. $(7 \cdot 1) + (7 \cdot 4)$ or $7 \cdot (1 + 4)$

Answers to Exercises 3-7b

1. a. $18 + 9 = 27$
   b. $3 \cdot 15 = 45$
   c. $5 + 56 = 61$
   d. $12 \cdot 8 = 96$
   e. $12 + 15 = 27$
   f. $7 + 15 = 22$

   Parts a, c, and f do not use the distributive property.

2. a. Commutative property for addition.
   b. Distributive property.
   c. Associative property for addition.
   d. Associative property for multiplication.
   e. Commutative property for multiplication.
   f. Distributive property.

3. 221,312

3-8. The Closure Property.

Emphasis here is placed upon the meaning of a set closed under an operation. The student is already familiar with the meaning of set and with the set of counting numbers and the set of whole numbers. Here is an excellent opportunity to review these concepts and point out once more the significant difference between the set of counting numbers and the set of whole numbers.

Good examples of sets closed under addition:

The set of whole numbers.

The set of counting numbers.

Then ask the class if these sets are closed under multiplication. Under subtraction. Under division.

Emphasize that if just one pair of counting numbers can be found such that their difference (or quotient) is not a counting number, then the set of counting numbers is not closed under subtraction (or division). For example, 9 - 12 is not a counting number for there is no counting number which can be added to 12 to get 9 and 12/9 is not a counting number since there is no counting number which can be multiplied by 9 to get 12. Since subtraction and division with two counting numbers are not closed, the need for negative numbers and rational fractions now becomes apparent.

The commutative, associative, distributive and closure properties and the identity properties of 0 and 1, all of which are encountered in this chapter, are very fundamental in Modern Algebra. These properties are part of the small list of axioms for high school Algebra from which everything else can be derived. This axiomatic approach to Algebra is a fairly recent development (dating back to the first decade of this century) and until recently only a very few people have been familiar with it. Because of this, until the last few years, students were first introduced to this approach to Algebra in a graduate course in the university.

It turns out that this allegedly sophisticated approach is actually easier than the conventional one in that it organizes and clarifies the subject. This method of presenting Algebra is employed at the ninth grade level in the various modern mathematics programs for the schools.

The axioms we are referring to are tabulated below.

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I. Field Axioms. A set of objects (numbers) R is called a field, if, whenever a, b, and c are in R we have:

	Addition	Multiplication
Closure	$a + b$ is in R	$a \cdot b$ is in R
Commutative	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative	$a + (b+c) = (a+b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Identity	There is a number 0 in R for which $a + 0 = a$	There is a number 1 in R for which $a \cdot 1 = a$
Inverse	There is a number $(-a)$ in R for which $a + (-a) = 0$	If $a \neq 0$ there is a number $\frac{1}{a}$ in R for which $a \cdot \frac{1}{a} = 1$
Distributive	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	

It is interesting to note that we obtain an equivalent set of axioms if we remove the Identity and Inverse Axioms and replace them by **the following solvability axioms:**

Solvable	There is a number x in R for which $a + x = b$	If $a \neq 0$, there is a number y in R for which $a \cdot x = b$

The "rules" that are taught in a traditional algebra course can be proved as consequences of these field axioms. For example, we can prove the following statement:

Proof:

$0 = a + (-a)$	Inverse (Addition)
$= a \cdot 1 + (-a)$	Identity (Multiplication)
$= a \cdot (1+0) + (-a)$	Identity (Addition)
$= [a \cdot 1 + a \cdot 0] + (-a)$	Distributive
$= [a + a \cdot 0] + (-a)$	Identity (Multiplication)
$= [a \cdot 0 + a] + (-a)$	Commutative (Addition)
$= a \cdot 0 + [a + (-a)]$	Associative (Addition)
$= a \cdot 0 + 0$	Inverse (Addition)
$= a \cdot 0$	Identity (Addition)

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II. **Order Axioms**. If in addition to the field axioms, R satisfies the axioms below, then R is called an ordered field. Whenever a, b, and c are in R we have:

Trichotomy	Exactly one of the following three statements is true: $a < b$, $a = b$, $b < a$.
Transitive Property	If $a < b$ and $b < c$ then $a < c$.
Addition Property	If $a < b$ then $a + c < b + c$.
Multiplication Property	If $a < b$ and $0 < c$ then $a \cdot c < b \cdot c$.

III. **Completeness Axiom**. If in addition to the field and order axioms, R satisfies a completeness axiom, then R is called a complete ordered field. The three possible forms of the completeness axiom given below are all equivalent in the light of the other axioms.

a. Dedekind Cut	If A and B are non-empty subsets of R which satisfy the condition: *(i) $A \cup B = R$, **(ii) $A \cap B$ is empty, (iii) each member of A is less than each member of B; then either there is a largest number in A or there is a smallest number in B.
b. Least upper bound	Every non-empty set of numbers which has an upper bound also has a least upper bound.
c. Infinite decimals	Every number has a unique representation as an infinite decimal having infinitely many digits different from 9.

* See Chapter 4 for a discussion of intersection of sets.
** See Chapter 7 for a discussion of union of sets.

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Only the third form of the completeness axiom is mentioned in this text.

The set of rational numbers satisfies the field and order axioms but not the completeness axiom. The real numbers satisfy all the axioms.

The way in which these axioms are used in this course follows this outline. The closure, commutative, associative, identity and distributive properties are observed in Chapter 3 to hold for counting numbers in a number of examples and are assumed to be true in general. (In effect they are assumed as axioms.) In Chapter 6 it is observed that the set of counting numbers is not closed under division--or in other words, that the solvability property of multiplication does not hold--and this is used to motivate the extension of our number system to embrace the non-negative rational numbers. Then it is shown that if we wish to retain the commutative, associative, identity and distributive properties we must multiply, divide, add and subtract rational numbers in just the way we always have. In Ch.17 (Vol.II) we show analogously how the lack of closure under subtraction--or the absence of the solvability property for addition--leads to the extension of our number system to embrace the full set of rational numbers, positive and negative and zero. In Chapter 20 it is observed that the real numbers have the completeness property while the rational numbers do not.

The following chart shows the chapters in which the various properties first come up for extensive discussion.

Field

	Addition	Multiplication
Closure	3	3
Commutative	3	3
Associative	3	3
Identity	3	3
Inverse	17	6
Distributive	3	

Solvability	17	6

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Order

	Addition	Multiplication
Trichotomy	8	
Transitive	8	
Addition	8	
Multiplication	8	

Completeness

Infinite Decimals	20

Answers to Exercises 3-8a

1. a. No. The sum of 2 odd numbers is always an even number.
   b. No.

2. a. M is the set of counting numbers which are divisible by 5.
   b. Yes. Since each of the numbers in the set is a multiple of 5, the sum of any two numbers in the set is a multiple of 5.
   Below is the general proof that set M is closed under addition:

   Let a and b represent any counting numbers.
   Then $5a$ and $5b$ must represent any two numbers in set M, and $5a + 5b$ represents their sum. Therefore $5a + 5b = 5(a + b)$ by the Distributive Property. Since one of the factors of the right member is 5, then the sum $5a + 5b$ must be a multiple of 5.

3. Each set is closed under multiplication.

4. a. Yes
   b. No. For example $500 + 501 = 1001$ and 1001 is not in the set.
   c. No. For example $3 + 47 = 50$ and 50 is not in the set.
   d. Yes. If the numerals of 2 numbers end in 0, then the sum of the numbers ends in 0.

5. a. Yes. b. No. c. No. d. Yes.

6. Yes. Multiplication of whole numbers is an abbreviated process for addition.

7. No. The student may give any number of examples. For instance, 1 - 2.

8. No.

9. No. For example, the result of 3 divided by 4 is not in the set.

10. No.

Answers to Exercises 3-8b

1. a. 8219 b. 1928

2. a. 19,997 b. 1179

3. a. 78,528 b. 450,954 c. 2,499,574

4. Two million, seventy thousand, three hundred fifty-one.

5. 72 cents.

6. a. greater than
    b. equal to
    c. less than
    d. not equal to
    e. times

3-9. Inverse Operations.

The basic concepts in this section are:

1. The meaning of inverse.
2. Addition and multiplication have the closure, commutative, and associative properties, while their inverses do not.

The meaning of inverse may be explained by giving an example. "I write on the chalkboard" may be stated as one actually writes "inverse" on the board. Then the teacher may say "the inverse of writing on the board is erasing the writing from the board." The board may actually be erased. It should be emphasized that the inverse operation *undoes* the first operation. Some pupils may think that the failure to do an operation is the inverse of the operation. For example, to the question "What is the inverse of singing?" the pupil may say "Not singing." But "not singing" does not *undo* the operation of singing as erasing the chalkboard *undoes* writing on the chalkboard. In this connection it is important to point out that some operations have no inverse.

Some discussion of $a \cdot x = b$ may be helpful to many students.

The following questions may be suggestive.

1. What operation is indicated by $a \cdot x$?
2. What operation will undo multiplication?
3. What is the inverse of multiplication?
4. To undo $a \cdot x$, do we divide $a \cdot x$ by a or $a \cdot x$ by x?
   (Since $a \cdot x$ means a times x we divide by a, the multiplier.)
5. How do we undo $3 \cdot 2$? (Divide 6 by 3.)
   How do we undo $8 \cdot 4$? (Divide 32 by 8.)
6. In terms of these symbols, can you define division?

An understanding of $a \cdot x = b$ will be helpful to the pupil as he studies percentage, and the equivalence of the two statements "$b \div a = x$" and "$a \cdot x = b$" will be of great importance in Chapter 6. Therefore, an emphasis on understanding the relationship between a, x, and b is not only desirable but necessary.

We used the device of the two machines to illustrate the inverse of multiplication by a as the unary operation of division by a. The following may be used to illustrate the general statement.

In general, suppose we have two machines that perform operation ① and operation ②. And suppose we hook them together, and observe that whenever we put a number in the first machine we get the same number out of the second machine.

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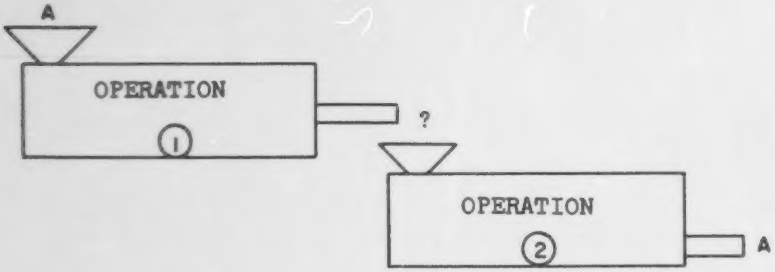

Then whatever was done by the first machine was undone by the second. We would say that operation ② is the inverse operation of operation ①.

Answers to Oral Exercises 3-9a

1. Laying down the pencil.
2. Take off your hat.
3. Get out of a car.
4. Withdraw your arm.
5. Division.
6. Tear down.
7. There is no inverse.
8. Step backward.
9. There is no inverse.
10. Subtraction.
11. Multiplication.
12. Addition.
13. There is no inverse.
14. There is no inverse.
15. Put a tire on a car.

Answers to Oral Exercises 3-9b

1. 5
2. 5
3. 1
4. 7
5. None
6. 0
7. 8
8. 4
9. 3
10. 3
11. None
12. 7
13. Any whole number
14. 4
15. 5
16. 5
17. 9
18. 9
19. 6
20. 0
21. 0
22. 0
23. None
24. 1
25. 1
26. 1

Answers to Exercises 3-9c

1. a. 46,471
   b. $507.10
   c. 506 feet
   d. $1,412.78
   e. $1,101.04
   f. $1,072.67
   g. 876
   h. 987
   i. 798
   j. 697

2. a. 12
   b. 31
   c. 3
   d. 5
   e. 588
   f. 463
   g. 3
   h. 6
   i. none
   j. none

3. a. 19
   b. 1992
   c. 89
   d. 19,219
   e. 165,821
   f. 13
   g. 6
   h. 20

4. a. 21
   b. 84
   c. 102
   d. 3
   e. 46
   f. 20
   g. 104
   h. 195

3-10. Betweenness and the Number Line.

The following understandings should be developed by the teacher so that the student will gain the fullest appreciation of betweenness and the order relations of numbers.

1. The number line helps to show how the counting numbers are related. The students may ask about the dots to the left of zero. The teacher may wish to mention that these are negative numbers and give a few illustrations, but the topic of negative numbers will not be discussed at all in our work.
2. A number is less than a second number if the first is to the left of the second. A number is greater than another if it is to the right of it.

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3. There is not always a counting number between two counting numbers. This fact is brought out in Exercises such as part 1(g) of Exercises 3-10.

4. To find the number of whole numbers between two other numbers (if it can be done at all): Subtract the smaller from the larger and then subtract one (1) from this difference. Or, subtract one (1) from the larger and then find the difference between that result and the smaller number. Or, add 1 to the smaller number and then subtract this result from the larger number.
Example: find the number of whole numbers between 7 and 15.

Method 1: 15 - 7 = 8 8 - 1 = 7
Method 2: 15 - 1 = 14 14 - 7 = 7
Method 3: 7 + 1 = 8 15 - 8 = 7

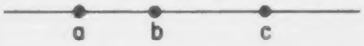
Answers to Exercises 3-10

1. a. 17
   b. 21
   c. 4
   d. 7
   e. None
   f. 2
   g. None
   h. 88

2. a. 10
   b. 11
   c. 24
   d. 30
   e. 18
   f. 22
   g. 16
   h. 9

3. The pairs of numbers in parts a, b, c, g, *i, and *j, have whole numbers midway between them.

*4. a. Yes, since c is to the right of a on the number line.
   b. Yes, since b is to the right of a on the number line.
   c. Yes, since b is to the left of c on the number line.
   The numbers a, b, and c would be located in the following manner on the number line.



3-11. The Number One.

In this lesson emphasis should be placed not only on special properties of the operations with 1 but also on the closure, associative, and commutative properties. The fact that there is more than one way to represent the number 1 is emphasized in the first exercise. Of course, this gives the teacher the opportunity to review the concept of numeral as a name for a number and not the number itself. Pupils think of the operation with numbers so frequently that they forget that $4 - 3$ is really another way to represent 1.

We note that 1 is not an identity for division. Since $a \cdot b = b \cdot a$ for all whole numbers, in particular 1 acts as an identity on each side. This is true for any commutative operation. However, division is not commutative. $a \div b \neq b \div a$. In particular $1 \div b \neq b \div 1$. $b \div 1 = b$ so that 1 is a right-handed identity for division.

A class discussion of the lesson summary in symbols may be profitable for many pupils. Of course, some other letter in place of c would be used as practice for pupils in translating symbols into words. Be sure that the pupils have the ideas before attempting symbolism. The pupils' translations of the mathematical sentences could be somewhat as follows:

a. If any counting number is multiplied by 1, the product is the same counting number.

b. If any counting number is divided by 1, the quotient is the same counting number.

c. If any counting number is divided by the same number, the quotient is 1.

d. If the number one is divided by any counting number, the quotient is one over the counting number.

e. The number one, raised to any power which is a counting number, equals 1.

Answers to Exercises 3-11

1. The symbols in the following parts represent the number 1:
   a, b, c, d, e, i, k, l, m, o, and p.
2. a. 100 b. 10 c. 14
   d. $\frac{2}{3}$ e. 0 f. 0

***3.** The successive addition of 1's to any counting number will give a counting number. But, the successive subtraction of 1's from any counting number will become 0 if the operation is carried far enough. If we go too far we get out of the set.

4. a. 876,429
b. 976,538
c. 897,638
d. 896,758
e. 3,479
f. 97
g. 1
h. 1

3-12. The Number Zero.

The purpose of this lesson is to understand why we can or cannot perform the fundamental operations with zero.

It is important for pupils to understand that zero is a perfectly good number and that it does not mean "nothing." The pupil should see that in addition and subtraction zero obeys the same laws as the counting numbers.

In explaining the product of $c \cdot 0$ and $0 \cdot c$, it may be helpful to review briefly the meaning of multiplication. Such discussion questions might be:

1. What is another way to find the answer to 3×5?
2. What does 3×5 mean? It means $5 + 5 + 5$ and not $3 + 3 + 3 + 3 + 3$.
3. Make up a real problem using 3×5. (The price of 3 pencils at 5¢ each.)
4. What does 5×3 mean? (It means $3 + 3 + 3 + 3 + 3$.)
5. Make up a real problem using 5×3. (The price of 5 pencils at 3¢ each.)

After such questions, zero may be introduced in the discussion as multiplicand and multiplier, since $5 \cdot 0 = 0 \cdot 5$ by the commutative property for multiplication.

In case of division, pupils should understand why we divide 0 by a and do not divide a by 0. It may be desirable to use several examples so that the pupils will see that $\frac{0}{c}$ (where c is a counting number) should be 0 and $\frac{c}{0}$ is not the name of any whole number.

Some of the pupils may be interested in why we do not define $\frac{c}{0} = 1$ or some other number. They should understand that it would be out of harmony with the fact that zero times any number equals zero.

The translations into words of the symbolic statements concerning zero can be somewhat as follows:

a. The sum of any whole number and zero is the same whole number.
b. If zero is subtracted from any whole number, the difference is the same whole number.
c. If any whole number is subtracted from itself, the difference is zero.
d. If c is a counting number, then 0 to the c-power is zero.
e. The product of any whole number and zero is zero.
f. If zero is divided by any counting number, the quotient is zero.
g. Zero cannot be used as a divisor.

Answers to Exercises 3-12

1. The symbols in the following parts represent the number 0: b, d, f, h, i, j, k, l, m, n, o, q, and s.

2. a. 3724
b. 73,788
c. 144 R 56
d. 152 R 60
e. $36,453
f. $60,444
g. 0
h. $846.25
i. $70.65
j. 679
k. 379
l. 897
m. $397.16
n. Division by zero is not possible.
o. 1
p. $1846
q. 0
r. 0
s. 0
t. 0
u. 976
v. $97.46

*3. The error is in the generalization to C in part i. If $a \cdot b = C$, a or b does not need to be C. Example: $2 \cdot 2 = 4$. This exercise shows the error that may be made by making a generalization on just a few cases.

4. The number is one or zero.

3-14. Chapter Review.

Answers to Exercises 3-14

1. The commutative property of multiplication is illustrated.
2. In the set of counting numbers the identity element for multiplication is one (1).
3. $2 \cdot 13 + 5 \cdot 13 = (2 + 5) \cdot 13$
4. $136 + 25 + 75 = 136 + (25 + 75)$
   $= 136 + 100$
   $= 236$
5. Multiply 65 times 11 and see if the product is 715. The division is correct.
6. The number of counting numbers between 6 and 47 is 40 .
7. Zero is a member of the set of whole numbers. Zero is not a member of the set of counting numbers.
8. The statement is false. If one can find one example where the operation is not closed, then the operation is not closed for the set of whole numbers.
9. The value of 1^{12} is 1 .
10. There are no counting numbers between 5 and 6.
11. $100 \div (20 \div 5) = 100 \div 4$
    $= 25$
12. The distributive property involves two operations: addition and multiplication.
13. a. 5
    b. Any whole number
    c. Any whole number larger than 2.
    d. 2
14. The identity element for addition of whole numbers is 0 .
15. No. For example, $6 + 9 = 15$, and 15 is not in the set.
16. a. The inverse operation of division is multiplication.
    b. The inverse operation of subtraction is addition.

17. 5 - 4, 7 ÷ 7, and 1 - 0 are different symbols which all represent the number 1.
18. a. 7 is greater than 2.
    b. 15 is less than 33.
    c. 4 is less than 6 and 6 is less than 10.

3-15. Cumulative Review.

Answers to Exercises 3-15

1. $(122)_{three} = (17)_{ten} = (32)_{five}$. (It is easiest to get to base five by going through base ten.)
2. Yes. Start by filling either the 3-cup or the 5-cup container. If the three-cup container is filled first then: (a) Pour 3 into 5; (b) Fill 3; (c) Pour 2 from 3 into 5, which leaves 1 in 3; (d) Empty 5, pour 1 left in 3 into 5; (e) Fill 3. Now we have 4. If the five-cup container is filled first then: (a) Pour 3 from 5 into 3, which leaves 2 in 5; (b) Empty 3 and pour in 2 from 5; (c) Fill 5; (d) Fill 3 by pouring 1 from 5; (e) Empty 3. Now we have 4.
3. 1,111
4. 4 × 4 × 4 × 4 × 4
5. Base 2
6. (2 × 27) + (0 × 9) + (1 × 3) + (0 × 1) = 57
7. 21
8. 100, 101, 102, 103, 104 (all base six numerals)
9. $(45)_{ten}$
10. Zero is an element of the set of whole numbers, but zero is not an element of the set of counting numbers.
11. Commutative property of multiplication.
12. Multiplication.
13. (3 + 2) · 5 or 5 · (3 + 2)
14. (125 + 75) + 36
15. Multiply 125 times 3 and see if the product is 375.
16. 37

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17. Subtraction

18. Division

19. a. eleven is greater than eight

    b. six does not equal ten

    c. two is less than four

Sample Test Questions for Chapter 3

1. Insert a symbol which makes a true statement:

   $8 + 4 \qquad 4 + 8.$

2. How many days are there between March 13, 1951 and March 27, 1951?

3. Show with one example that the set of numbers from 10 to 15 is not closed under addition.

4. Answer true or false: The identity for multiplication in the set of whole numbers is zero.

5. If K is a counting number then $\frac{0}{K} = ?$

6. Apply the commutative property of addition to: $(4 \cdot 5) + 6.$

7. We are using the _______ property when we say that $3a + 5a$ is another way of writing $(3 + 5) \cdot a$.

8. If the product of 5 and a certain number is zero, then that number must be:

   a. 1 b. 0 c. 5 d. None of the above.

9. When the number one is divided by any counting number n, the answer is always:

   a. 0 b. 1 c. n d. None of the above

10. Which of the following numerals are names of counting numbers?

    $(10)_{two}$ 1^4 0 $(713)_{ten}$ $\frac{2}{3}$ XIV

11. $[(7 + 3) + 7 \cdot (6 \cdot 5)] = [(7 + 3) + (7 \cdot 6) \cdot 5]$ illustrates the associative property of __?__ .

12. How many counting numbers are there between $(10)_{five}$ and $(13)_{four}$?

13. Make a true statement of $(3 \cdot 7) + (4 \quad) = 7 \cdot (\quad)$.

14. To check the statement $7 \times 345 = 2415$ by the inverse operation we would __?__.

15. The letters a, b, and x represent counting numbers, and $\frac{a}{b} = x$. What can we say about the relation between a and b?

Answers to Sample Test Questions

1. =
2. 13
3. $12 + 13 = 25$
4. False
5. 0
6. $6 + (4 \cdot 5)$
7. Distributive
8. b
9. d
10. 10_{two}, 14, 713_{ten}, XIV
11. Multiplication
12. 1
13. $(3 \cdot 7) + (4 \cdot 7) = 7 \cdot (3 + 4)$
14. Divide 2415 by 7 (or by 345)
15. a is a multiple of b, and a is either greater than b or equal to b.

Chapter 4

NON-METRIC GEOMETRY I

Some remarks are in order to explain the purposes of this chapter. In the conventional 10th grade course geometry is presented as a mathematical theory in which theorems are proved from axioms using rules of logic. What is being attempted here is something quite different but it would be a grave error to look upon our approach with disdain. The approach in this chapter is a very necessary prelude to the axiomatic development.

Everyone has an intuitive grasp of the concepts of point, line, and plane and everyone will agree that certain elementary properties pertaining to these concepts are obvious. Many efforts have been made to define these elementary concepts and to prove these elementary properties but these efforts seem most unsatisfactory, belonging more to the realm of metaphysics than of science. The reason that these attempts are so unsatisfactory is that the problems are unsolvable. The process of definition is the statement of the meaning of new words or expressions in terms of words or expressions where meaning is already known. The process of deductive proof is the establishing through the rules of logic of the truths of statements by use of statements already known to be true.

When these facts are recognized, then we see that any body of knowledge must contain certain undefined terms and certain statements which are accepted as true without proof. "Deductive science" and "mathematical theory" are names applied to bodies of knowledge in which the undefined terms and statements to be accepted without proof (axioms) are clearly set forth at the outset. Thereafter all other terms used are defined by means of these undefined terms and any other statements accepted as true must have been proved first by logical reasoning from the axioms and statements (theorems) previously established. A mathematical theory such as geometry is, then, a collection of statements about certain undefined concepts derived logically from certain unproved statements.

The connection with "reality" depends on common acceptance of the axioms. If a person accepts the axioms as properties which are true of his concepts of points, lines, and planes then he must accept the rest of the theory--the logical consequences of these axioms. The selection of the axioms is achieved by the process of inductive reasoning applied to the intuitive concepts of point, line, and plane. After the axioms have been selected, inductive reasoning drops from sight in the formal structure of the theory where only deductive reasoning is used. Inductive reasoning nevertheless retains an important place in the creative process. The creative process in mathematics consists of two parts: first, the conjecture or guess that a certain statement is true; second, the proof of the statement. The conjecture is made by use of intuition and inductive reasoning. The proof is accomplished by deductive reasoning.

The main purpose of geometry in this text is to present intuitively the concepts of point, line, and plane and to reach agreement by inductive reasoning that certain statements concerning these concepts appear to be true. Some of these statements will appear in the formal geometry course as axioms. Others will be proved as theorems. A second purpose of geometry in this book is to present an introduction to the process of deductive reasoning in geometry.

The principal objectives of this chapter are threefold:

1. To introduce pupils to geometric ideas and ways of thought,
2. To give pupils some familiarity with the terminology and notation of "sets" and geometry, and
3. To encourage precision of language and thought.

There is an attempt to guide the student to the discovery of unifying concepts as a basis for learning some of the more specific details. This chapter forms a background for later chapters which deal with metric or distance properties. It attempts to focus attention upon ideas which are fundamental but which (while sometimes vaguely taken for granted) are often poorly understood by students.

Traditionally, those ideas have been taught as they were needed for a particular geometric discussion. But, all too often, the teacher has assumed that these properties are obvious or clear without mentioning them. Also, there should be some advantage in considering together this group of closely related analogous properties and observing relations among them. The higher level study of some aspects of non-metric geometry has become a separate mathematical discipline known as projective geometry.

Spatial Perception

One of the important aims of this chapter is to help boys and girls to develop spatial imagination. There is some basis for the belief that slower students, in many cases, have powers of visualization that compare favorably with those possessed by more able students. To what extent there is a correspondence between general intelligence and spatial understanding is difficult to determine. Therefore, it would be desirable for the teacher to exploit the interest shown by the slower student in devising drawings or other representations of spatial relations. If given free rein, students will devise ingenious models for spatial representations.

Time Schedule

A definitive time schedule may be inadvisable. The variations among pupils may be so great that a flexible schedule is advised. For general guidance to the teacher the following schedule is indicated. The teacher must not feel that this is to be followed rigidly.

Lessons 1-5 -- Sections 1 and 2
Lessons 6-12 -- Rest of Chapter

As to the specific ground to be covered in each lesson, the teacher must use his own judgment. An attempt has been made to include exercises at frequent intervals. In some cases, a subsection together with the appended exercises will make a satisfactory unit lesson, but this will not always be the case.

Materials

Insights into ideas developed in this chapter will be greatly enhanced by use of instructional devices. Encourage students to make simple models as a means of developing basic understandings. Emphasize ideas, not evaluation of models. The use of a tinker toy set or D-sticks will be found helpful in representing spatial relations. In using any instructional material of this kind, seek understanding of ideas without over-dependence upon representations.

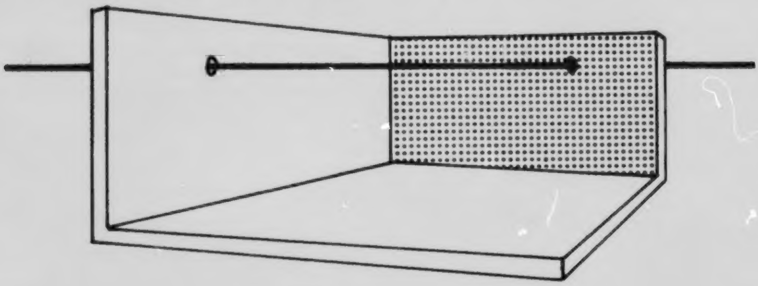
Suggested Materials

STRING--to represent lines in space.

PAPER--to represent planes, and folded to represent lines and intersections of planes.

TAPE OR TACKS--for attaching string to walls, floor, and points in the room.

MODEL--(as illustrated)



Suggest making the model as shown above by using a cardboard carton (or, it can be made using heavy paper, oak tag, screen wire). Cut away two sides so that only two adjacent sides and bottom of the box remain. String, wire, etc., may be used to extend through and beyond "sides," "floor," etc.

OAK TAG--for making models to be used by both teacher and students.

COAT-HANGER WIRE, KNITTING NEEDLES, PICKUP STICKS

SCISSORS, COLORED CHALK.

LIGHTWEIGHT PAPER--for tracing in exercises.

YARDSTICK or meter stick with several lengths of string tied to it at different intervals. By fastening stick to wall, lines may be represented by holding the string taut. By gathering together the free ends at one point the plane containing the point and the yardstick may be shown.

OPTIONAL

Long pointer for indicating lines.

Toothpicks for student models

Saran wrap, cellophane, and wire frame for representing planes.

4-1. Points, Lines, and Space.

1. Understandings.
   a. A point has no size.
   b. A line is a certain set of points.
   c. A line is unlimited in extent.
   d. Through two points there is one and only one line.
   e. Space is a set of points.
2. Teaching Suggestions.

Just as we use representations to develop the concept of the "counting numbers" (2 cars, 2 people, 2 hands, 2 balls, 2 chairs, etc., to develop the concept of twoness) similarly we must select representations for developing the concepts of point, line, plane, and space.

Point

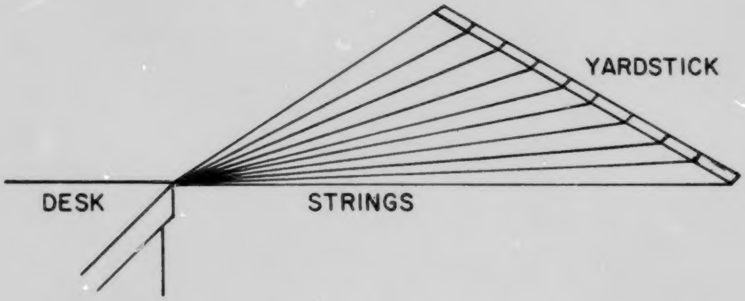
Identify things which suggest the idea of a point keeping in mind that one suggestion by itself is inadequate for developing the idea of a point. One needs to use many illustrations in different situations. Avoid giving the impression that a point is always identified with a tip of a sharpened object. Suggestions: pupil of the eye in intense brightness, dot of light on some TV screens, particle of dust in the air.

Line

Identify two points using some of the situations as above. Bring out the idea that given these two points there are many other points of the line that contain them. Some of these are between the two points, some are "beyond the one" and some are "beyond the other." Also, through two points there can be only one line. The line has no thickness and no width. It is considered to extend indefinitely. Use string held taut between two points to show representations of lines in positions that are horizontal, vertical, and slanting. Each student may represent lines by using a pencil between his fingertips. With each example, talk about thinking of a line as unlimited in extent. Emphasize frequently that we use the word "line" to mean straight line. Identify other representations of lines such as: edge of tablet (holding the tablet in various positions), edge of desk, vapor trails, edge of roof of building, etc. It is important to select illustrations representing lines in space as well as the usual representations made by drawing on chalkboard and paper.

Space

Models will be most helpful here. Using "string on yardstick" and considering some point on a table, desk, or on some object which all students can see, let all the representations of lines from the yardstick pass through the point. Also, use string to show representations of other lines from other points on different walls, the floor, etc., all passing through the point. Use the model as described in drawing under "Suggested Materials." Pass lines (string, wire, thread) through "walls" and "floor" to suggest infinite number of lines and that these lines extend indefinitely. Bring out the idea that each line is a set of points, and that space is made up of all the points on all such lines.



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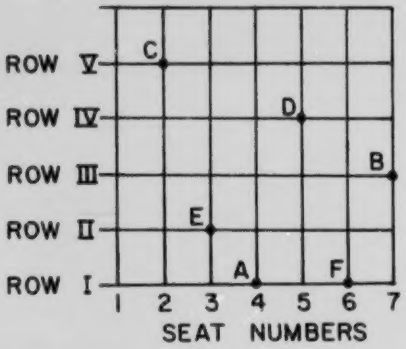
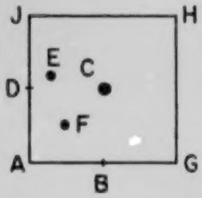
Answers to Exercises 4-1a

1. Depends upon ingenuity of students in finding objects which represent points.
2. a. T
   b. P
   c. S
   d. D
3.



ROW V
ROW IV
ROW III
ROW II
ROW I
C
D
B
E
A
F
1 2 3 4 5 6 7
SEAT NUMBERS

4. a. H b. E c. B d. N
5.

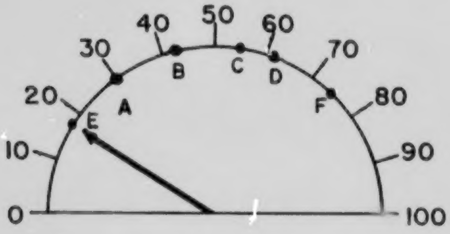


The dimensions of regulation baseball and softball fields are:

Baseball: 90' square infield, homeplate to pitcher's mound 60'6", homeplate to 2nd base 127' $3\frac{3}{8}$".

Softball: 60' square infield, homeplate to pitcher's mound 46'.

6.



0
10
20
30
40
50
60
70
80
90
100
E
A
B
C
D
F

page 118-120: 4-1, 4-2

Answers to Class Exercises 4-1b

1. 3, $\overleftrightarrow{AB}$, $\overleftrightarrow{FG}$, $\overleftrightarrow{HD}$

   In naming the lines encourage students to use any two letters that will identify the line.
2. $\overleftrightarrow{AC}$ and $\overleftrightarrow{KF}$, B etc.
3. No. The crepe paper ribbons do not form straight lines.
4. a. A, B, or C
   b. $\overleftrightarrow{SV}$ or $\overleftrightarrow{WX}$
   c. $\overleftrightarrow{DE}$
   d. $\overleftrightarrow{RT}$ (other names are $\overleftrightarrow{RL}$ or $\overleftrightarrow{LT}$)
   e. $\overleftrightarrow{KL}$ and $\overleftrightarrow{RT}$, L etc.

Answers to Exercises 4-1c

1. Depends upon classroom.
2. The porcupine has quills which suggest lines emanating from the body. If we consider the body a point, then space is like the set of points consisting of all the points on all the lines (quills).
3. Depends upon gymnasium.
4. The boundary lines are fixed because the points determining these lines are fixed. Property 1 states: Through any two different points in space there is exactly one line.

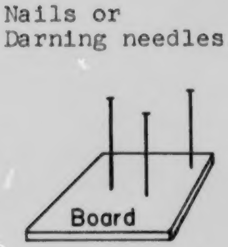
4-2. Planes

1. Understandings.
   a. A plane is a set of points in space.
   b. If a line contains two different points of a plane, it lies in the plane.
   c. Many different planes contain a particular pair of points.
   d. Three points not exactly in a straight line determine a unique plane.
2. Teaching Suggestions.

   Identify surfaces in the room which suggest a plane--

walls, tops of desks, windows, floor, sheet of paper, piece of cardboard, chalkboard, shadow. Make use of Saran wrap, cellophane, and a wire frame to show further a representation of a plane since this more nearly approaches the mathematician's idea of a plane. With each example bring out the idea that a plane has no boundaries, that it is flat, and extends indefinitely. It is an "ideal" of a situation just as are a line and a point. We try to give this idea by suggesting things that represent a plane. It is important to suggest representations of planes in horizontal, vertical, and slanting positions. Note that if a line contains 2 points of a plane, it lies in the plane and that many planes may be on a particular pair of points as pages of a book, revolving door, etc.

Then using three fingers or sticks of different heights in sets of 3 (not in a straight line) as suggested by the sketch at the right, see what happens when a piece of cardboard is placed on them. Add a fourth finger or a fourth stick and observe what happens.



Each student may try this experiment by using three fingers of one hand (and also three fingers using both hands) letting a plane be represented by a book, piece of oak tag, or card. Change the position of the fingers and thumb by bending the wrist (changing the sticks in the model). Ask the class to make a statement about three points not in a straight line. (Property 3.)

Demonstrate with wires or string the ideas in the last paragraph before asking the students to read it or suggest that one or two students be responsible for demonstrating the idea to other members of the class.

The Class Discussion Problems may well be developed as a class activity.

A note. What is a basic motivation for the study of geometry? In our daily living we are forced to deal with many flat surfaces and with things like flat surfaces. It would be foolish not to note similarities of these objects, so, we try to note them. In so doing we try to abstract the notion of flat surface. We try to find properties that all flat surfaces have. Thus, we are led to an abstraction of the flat surface-- the geometric plane. We study two aspects of this.

1. What a plane is like, considered by itself (plane geometry), and
2. How various planes (flat surfaces) can be related in space (one aspect of spatial geometry).

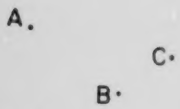
Just how do we study the geometric plane? We study it by thinking of what the plane is supposed to represent, namely, a flat surface. However, in trying to understand a plane (or planes) we find it difficult to think in the abstract. Thus, we think of representations of the plane: wall, chalkboard, paper, etc., and we think of these as representations of the abstract idea. The abstract idea enables us to identify characteristics which all flat surfaces have in common.

Answers to Exercises 4-2a

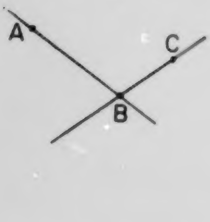
1. Depends upon the particular kitchen. Most kitchens would probably have the following examples: tabletop, shelves, seats of chairs, blades of knives, bottoms of pots, pans, and baking dishes in addition to floor, ceiling, and walls.
2. Would depend upon particular library.
3. Consult dictionary.

Answers to Class Exercises 4-2b

1.

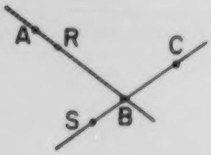


2.



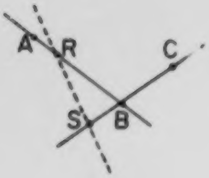
Yes. If a line contains two different points of a plane, it lies in the plane. (Property 2.)

3.



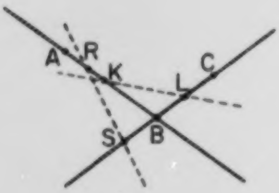
Notice that there are many different locations for points R and S. You may want to review the idea of betweenness by asking various students where they placed their points. (Property 2.)

4.



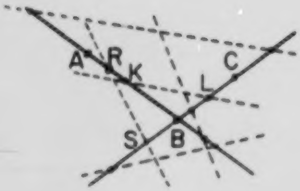
If a line contains two different points of a plane, it lies in the plane. (Property 2.)

5.



Same reason as step 4.

6.



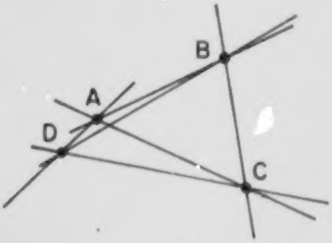
This figure is one possibility. There are others.

Answers to Exercises 4-2c

1. Would depend upon the particular classroom.
2. Any three of these points not exactly in a straight line are in one plane. Thus, there will be 4 planes--XYZ, XYW, XZW, YZW.

3. The feet of a three-legged stand are in only one plane. The addition of a fourth point creates the possibility of having 4 different planes. Refer to previous problem. A three-legged stand will therefore maintain one position. A four-legged stand may take any one of 4 different positions and is therefore unstable unless constructed so that the 4 points are in the same plane.
4. a. Many planes. If we have 2 points, then many planes contain these points.
   b. Only one line. Through any two different points in space there is exactly one line.
   c. One plane. Through any three points, not all on the same line, there is exactly one plane.
5. $\overleftrightarrow{AB}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BD}$. If a line contains two different points of a plane, it lies in the plane.
6. a. 6
   b. 6

A simple model to explain this problem can be made with cardboard, elastic thread, and thumbtacks.



A, B, C, and D are tacks, representing the four points. Stretch elastic thread between the points to represent the 6 lines. Merely lift one of the tacks off the cardboard to illustrate part b. The elastic keeps the lines straight and obviously no new lines are possible.

4-3. Names and Symbols.

1. Understandings.

Note: Notation for naming points and lines was introduced in 4-1 in order to facilitate class discussion. This is reviewed and simplified in 4-3.

a. Students learn to recognize how planes, lines on planes, lines through planes, etc. can be represented by drawings.

b. Students learn to name particular points, lines, and planes, using letters, etc.

c. Students learn how to interpret and understand perspective drawings.

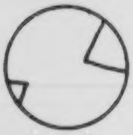
d. Students learn to develop an awareness of planes and lines suggested by familiar objects.

2. Teaching Suggestions.

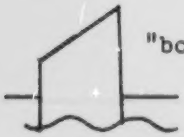
Reinforce the idea that we make agreements as to how to represent certain ideas i.e., " . " for a point, "—" or "$\overleftrightarrow{\quad}$" for a line, and the use of letters for naming lines and points. We usually name points by capital letters, lines by lower case letters or by pairs of capital letters with bar and arrows above, as $\overleftrightarrow{AB}$. A plane is named by three capital letters. A plane may also be named by a single capital letter although this convention may cause some confusion with slow pupils. It is generally avoided in this chapter. Also, we sometimes talk about two or more lines, planes, etc., by using subscripts, such as ℓ_1, ℓ_2, and ℓ_3.

We do not expect students to learn to make drawings showing more than one plane, intersections of planes, etc. Some students, however, may have considerable talent in this direction. Such students should be encouraged to make drawings which the entire class may find useful. For the class as a whole, the emphasis should be on the interpretation of drawings.

Students enjoy a guessing game about abstract figures such as these:

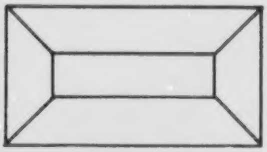


"looking out of an opened soda pop can.



"boat sinking"

They might enjoy a similar game with planar abstractions:



"big picture frame for small picture"   "bird's eye view of house roof"

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Answers to questions in 4-3: $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ lie in the plane.... Other names for plane ABE are plane AEC and plane BCE.

Answers to Exercises 4-3

1. a. ABC, or ADC, or BCD, etc.
   b. $\overleftrightarrow{AB}$, or $\overleftrightarrow{BC}$, or $\overleftrightarrow{CD}$, or $\overleftrightarrow{AD}$.
   c. For $\overleftrightarrow{AB}$, plane ABF or plane ABD, etc.
   d. FBC, FGC, etc.
2. It has turned upside down.
3. a. cot
   b. pingpong table
   c. football field
   d. carpet
   e. high jump
   f. coffee table
   g. line of laundry
   h. open door
   i. chair
   j. shelf
   k. ladder
5. Yes. No. No. One.
6. 1. b.
   2. d.
   3. c.
   4. d.
   5. a.
   6. a.

   Stress that the second column suggests an advantage of the subscript way of labeling.
7. a. yes
   b. yes
   c. yes
   d. ℓ_2
   e. no
   f. no, no
8. a. ABC or BCD or ABD
   b. CDF or CEF or DEF
   c. $\overleftrightarrow{GB}$
   d. HBC and ABC, etc.

page 133-134: 4-4

4-4. Intersection of Sets.

1. Understandings.
   - a. A set usually contains elements which are collected according to some common property or explicit enumeration.
   - b. The common elements in two or more sets make up the elements of the intersection of two or more sets.

2. Teaching Suggestions.

Review the idea of sets by asking students to describe certain sets, as set of names of members of the class, set of all students in the class whose last name begins with "B", set of even numbers, set of counting numbers between 12 and 70 having a factor 7 (i.e., {14, 21, 28, . . .}).

Explain that any two sets determine a set which is called their intersection, that is, the set of elements (if any) which are in both sets. Have students give the intersection for the set of odd numbers between 1 and 30 and the set of counting numbers having the factor 3 between 1 and 30. Note the three sets: the two given sets and the intersection of the two sets. Use other illustrations such as the set of boys in the class and the set of students with brown eyes. In selecting sets, include some in geometry (i.e., the intersection of two lines in the same plane, etc.). Note that the empty set is the intersection of two sets with no elements in common.

After developing the idea of intersection go back to examples and describe how the idea can be expressed in symbols. It is a code we can use and like many codes it simplifies the expression. For example,

Set $A = \{3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29\}$

Set $B = \{3, 6, 9, 12, 15, 18, 21, 24, 27\}$

$A \cap B = \{3, 9, 15, 21, 27\}$

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Answers to Exercises 4-4a

1. a. {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}.
   b. Would vary with student.
   c. Would vary with student.
   d. The empty set.
   e. {18, 19, 20, 21, 22} Stress the meaning of "greater than" and "less than."
2. a. Any three states, such as, California, Iowa, Maine.
   b. Any three months, such as March, May, July.
   c. Numbers such as 5, 10, 15, ...
3. a. {1, 9}
   b. {John, Frank, Alice}
   c. {9, 10, 11, 12}
   d. Would vary with student.
   e. {July 4}
   f. The empty set.
4. a. {9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}
   c. {1,2, 3, ...}
   d. {P}

The latter portion of this section provides introductory and exploratory material on intersections of geometric figures. These ideas will be developed formally in the next section.

Answers to Exercises 4-4b

1. a. Any three labeled points such as P, Q, R, etc.
   b. Any three lines such as, $\overleftrightarrow{AD}$, $\overleftrightarrow{AB}$, $\overleftrightarrow{AJ}$, etc.
   c. Any three planes such as, ABC, BCG, HFG, etc.
   d. Any three intersections of planes such as $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{EF}$, etc.
   e. Any three intersections of lines such as, points B, C, D, etc.
2. a. {G}
   b. {D}
   c. {A}
   d. The empty set.

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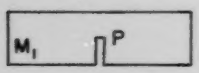
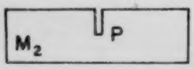
3. a. $\overleftrightarrow{HK}$
   b. point L
   c. the empty set
   d. point P

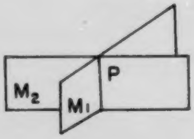
4-5. Intersections of Lines and Planes.

1. Understandings.
   a. Two lines may:
      (1) be in the same plane and intersect.
      (2) be in the same plane and not intersect (intersect in the empty set).
      (3) be in different planes and not intersect (intersect in the empty set).
   b. A line and a plane may:
      (1) not intersect (intersect in the empty set).
      (2) intersect in one point.
      (3) intersect in a line.
   c. Two different planes may:
      (1) intersect and their intersection will be a line.
      (2) not intersect (have an empty intersection).
2. Teaching Suggestions

Use models in order to explore the possible situations for two lines intersecting and not intersecting. (Let each student have materials, too.) Also, use a pencil or some other object to represent a line, and a card to represent a plane. Use two pieces of cardboard each cut to center with the two fitted together to represent the idea of two planes and their intersection, and, from these, state some generalizations that may be made.







page 137-139: 4-5

Also, identify situations in the room which are representations of different cases of intersections of lines and planes. Some may wish to express the ideas in symbols.

Subscripts also may be used to talk about lines ℓ_1 and ℓ_2 The use of a few subscripts should be encouraged. The students have been prepared for this in the previous section.

Answers to Questions in 4-5

1. $\overleftrightarrow{AD} \cap \overleftrightarrow{CD} = D$, $\overleftrightarrow{GF} \cap \overleftrightarrow{FE} = F$, $\overleftrightarrow{AB} \cap \overleftrightarrow{BC} = B$, etc.
2. $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ or $\overleftrightarrow{GF}$ and $\overleftrightarrow{HE}$

Property 3a. The questions in the text that precede the statement of this property constitute an informal proof. The teacher should be sure to understand the logical argument before questioning the class.

> A, B, and C are in exactly one plane since through any three points, not all on the same line, there is exactly one plane. (Property 3)
>
> $\overleftrightarrow{AB}$ lies in this plane since, if a line contains two different points of a plane, it lies in the plane. (Property 2) $\overleftrightarrow{AC}$ lies in this plane for the same reason.

Answers to Exercises 4-5a

2. a. $\overleftrightarrow{AD} \cap \overleftrightarrow{DC} = D$, etc.
   b. $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$, etc.
   c. $\overleftrightarrow{AB}$ and $\overleftrightarrow{KJ}$, etc.
3. CD and CG are intersecting lines. If two different lines intersect, exactly one plane contains both lines.
4. Points A and D are on $\overleftrightarrow{AD}$. Points D and C are on $\overleftrightarrow{DC}$. Through any two different points in space there is exactly one line. $\overleftrightarrow{AD}$ and $\overleftrightarrow{DC}$ intersect. If two different lines intersect, exactly one plane contains both lines.
5. a. Planes ABC, ABJ, CDK, EFA.
   b. $\overleftrightarrow{ED}$ and $\overleftrightarrow{BA}$, $\overleftrightarrow{EF}$ and $\overleftrightarrow{BA}$, $\overleftrightarrow{BJ}$ and $\overleftrightarrow{CD}$, etc.
   c. Points G, H, J, K all lie in the same plane.

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In discussing the intersection of two planes in class, do not stress the reasoning that leads to the conclusion that two planes intersect in a line. In fact, with many classes it would be desirable to show illustrations of this idea in the environment and omit the development of the reasoning.

Answers to Oral Exercises 4-5b

1. c.
2. c. (Notice that this question is the same as 1.)
3. a. (Parallel lines)
4. a. (Skew lines)
5. d.
6. a.
7. f.
8. f.
9. g.
10. a. (Parallel planes)
11. i.
12. i.

Answers to Exercises 4-5c

1. a. Plane ABE, plane FDC, plane EBC, plane EAT, plane ABC.
  b. $\overleftrightarrow{EB}$ and $\overleftrightarrow{AD}$, $\overleftrightarrow{EF}$ and $\overleftrightarrow{AB}$, etc.
  c. $\overleftrightarrow{EF}$ and $\overleftrightarrow{BC}$, $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$, etc.

2. a. Plane EAB and plane FDC.
  b. Plane EAB $\cap$ plane ABC = $\overleftrightarrow{AB}$, plane EAD $\cap$ plane ABC = $\overleftrightarrow{AD}$, etc.
  c. $\overleftrightarrow{AE} \cap \overleftrightarrow{EB} = E$, $\overleftrightarrow{AB} \cap \overleftrightarrow{BC} = B$, etc.
  d. Point A

3. a. Point E
  b. Point F
  c. The empty set
  d. $\overleftrightarrow{BC}$
  e. $\overleftrightarrow{AB}$
  f. The empty set
  g. The empty set

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4. a. Point V
   b. Point W
   c. The empty set
   d. The empty set
   e. $\overleftrightarrow{RT}$
   f. $\overleftrightarrow{TW}$
5. $\overleftrightarrow{RS}$ and $\overleftrightarrow{RT}$, plane SRT
   $\overleftrightarrow{ST}$ and $\overleftrightarrow{TW}$, plane XWT
   $\overleftrightarrow{RS}$ and $\overleftrightarrow{XR}$, plane RSX, etc.
6. a. $\overleftrightarrow{JE}$
   b. $\overleftrightarrow{AB}$ and $\overleftrightarrow{FL}$, $\overleftrightarrow{CB}$ and $\overleftrightarrow{DF}$, etc.
   c. The empty set
   d. Point J
   e. Plane ABL, or ABH, or ABJ, etc.
   f. Point E
   g. Point E
7. a. Plane HGD and plane ABC, etc.
   b. Plane HGB and plane GBC, plane FGD and plane FHE, etc.
   c. Planes BAH, BGC, and ABC, etc.
   d. Planes HGD, FGD, and BGD, etc.
   e. $\overleftrightarrow{GD}$ and plane ABC, $\overleftrightarrow{FE}$ and plane HGD, etc.
   f. $\overleftrightarrow{FE}$ and $\overleftrightarrow{GD}$, $\overleftrightarrow{AH}$ and $\overleftrightarrow{GB}$, etc.
   g. $\overleftrightarrow{HG}$ and $\overleftrightarrow{DC}$, $\overleftrightarrow{ED}$ and $\overleftrightarrow{AB}$, etc.
   h. $\overleftrightarrow{HG}$, $\overleftrightarrow{FG}$, and $\overleftrightarrow{GD}$; $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{BG}$, etc.
   i. Planes FGB, FGD, HGD, and BGD.

4-7. Chapter Review.

Answers to Exercises 4-7.

1. a. $\overleftrightarrow{ZN}$
   b. F, E, H, or G.
   c. The empty set.
   d. Point X.
   e. $\overleftrightarrow{EF}$ or $\overleftrightarrow{GH}$.
   f. Point C.
   g. The empty set.

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2. There are 4 **planes** as follows: plane ABC, plane ABD, plane ACD, plane BCD.
3. a. Plane ABD, plane CEF, plane EGH, plane XYZ.
   b. Planes ABD and CEF, the intersection is $\overleftrightarrow{CD}$. There are others.
   c. $\overleftrightarrow{AC}$ and $\overleftrightarrow{EF}$, $\overleftrightarrow{EG}$ and $\overleftrightarrow{DF}$, etc.
   d. Point D.
   e. The empty set.
   f. Point F.
   g. The empty set.
   h. The empty set.
4. $\overleftrightarrow{RT}$, $\overleftrightarrow{RV}$, $\overleftrightarrow{TV}$. If a line contains two different points of a plane, it lies in the plane.
5. a. Plane ADE.
   b. Plane ABD, or plane GFE.
   c. The empty set.
   d. $\overleftrightarrow{BG}$
   e. Point F.
   f. Point C.
   g. Point B.
   h. The empty set.
   i. The empty set.
   j. Only one line. Through any two different points in space there is exactly one line.
   k. Many planes, for example plane ADE, DHC, etc.
   l. Plane ABC and Plane DCF.

4-8. Cumulative Review.

Answers to Exercises 4-8

1. 8
2. a. two thousand, forty-two.
   b. thirty-seven thousand, two hundred fifty-six.
3. a. $(3 \times 10^3) + (4 \times 10^2) + (0 \times 10) + (7 \times 1)$
   b. $(1 \times 2^4) + (1 \times 2^3) + (1 \times 2^2) + (1 \times 2) + (1 \times 1)$
   c. $(2 \times 5^3) + (1 \times 5^2) + (4 \times 5) + (3 \times 1)$

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4. a. 81
   b. 81
   c. 64

5. Yes.

6. $2^5 = 32$, $5^2 = 25$, 2^5 is larger than 5^2.
   32 is 7 more than 25.

7. Base 4.

8. one.

9. 1; 1 raised to any power is another name for 1.

10. The set of multiples of 3 which are greater than zero.

11. Yes.

12. a. $9\cdot 3 + 9\cdot 2$
    b. $7\cdot 6 + 11\cdot 6$

13. All counting numbers are whole numbers.

14. a. $6 < 8$
    b. $9 < 12$ or $12 > 9$
    c. $3 > 0$
    d. $13 < 15 < 17$

15. $31 \times 5 \times 2 = 31 \times (5 \times 2)$

16. a. $\{6, 12, 18, 24\}$
    b. The intersection of these two sets has no elements.
       It is the empty set.

17. a. Infinitely many.
    b. one
    c. one

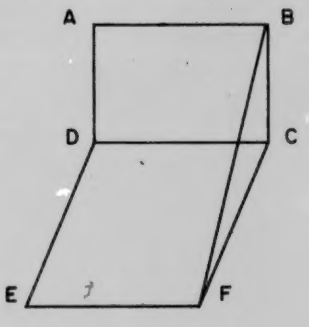
18. a. 1
    b. Four, ABC, ABE, BCE, ACE
    c. Point H.
    d. $\overleftrightarrow{ED}$
    e. The empty set.

Sample Test Questions for Chapter 4

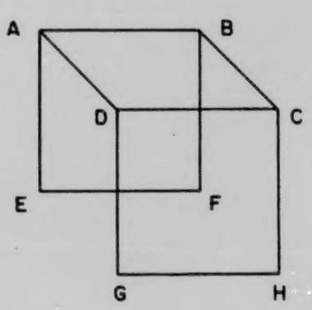
1. Suppose the intersection of two lines is the empty set. If the lines are in the same plane they are (parallel). If they are not in the same plane they are (skew).
2. Through any two different points in space, how many
   a. lines are there? (one)
   b. planes are there? (many)
3. If A, B, C, and D are four different points in space, no three of which are in a straight line:
   a. In how many planes do these points lie? Name the planes. (4 - ABC, ABD, ACD, BCD).
   b. In how many of these planes does the line $\overleftrightarrow{BC}$ lie?

4. If two different planes each contain the same three points, what can you say about the three points? (the points are on a line.)

5. In the figure at the right
   a. What is $\overleftrightarrow{CD} \cap$ plane BCF? (point C)
   b. What is plane BCF $\cap$ plane EFD? ($\overleftrightarrow{CF}$)
   c. $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ (the empty set)
   d. $\overleftrightarrow{AB} \cap$ is the intersection of plane ABC and plane _____. (plane ABF)

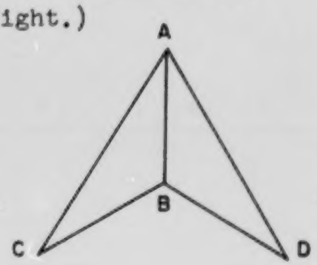


6. In the figure at the right
   a. What is plane ABF $\cap$ plane CDG? (the empty set)
   b. What is $\overleftrightarrow{CD} \cap$ plane CDG? ($\overleftrightarrow{CD}$)
   c. What is plane ABF $\cap$ plane ABC? ($\overleftrightarrow{AB}$)
   d. What is plane ADG $\cap$ plane BFH? (the empty set)



7. In the corresponding blank to the left of each of the following statements indicate if it is true or false.
   a. (False). The intersection of a line and a plane must be a point.
   b. (True). If the intersection of two planes is not the empty set then the intersection is a line.
   c. (True). A great many different planes may contain a certain pair of points.
   d. (True). If two different lines intersect, one and only one plane contains both lines.
   e. (False). Skew lines are lines that do not intersect.

8. Multiple Choice. (Use drawing at the right.)



   a. $\overleftrightarrow{AB}$ is the intersection of
      1. $\overleftrightarrow{CB}$ and $\overleftrightarrow{BD}$
      2. $\overleftrightarrow{AC}$ and $\overleftrightarrow{AD}$
      3. plane ABC and plane ABD
      4. A and $\overleftrightarrow{BC}$
   b. Points C, B, and D lie in
      1. plane ABC
      2. plane ABD
      3. $\overleftrightarrow{CB}$
      4. none of these.
   c. Plane ABD $\cap$ $\overleftrightarrow{AB}$ is
      1. Point B
      2. Point A
      3. $\overleftrightarrow{AD}$
      4. $\overleftrightarrow{AB}$
   d. $\overleftrightarrow{AC} \cap \overleftrightarrow{AD}$ is
      1. $\overleftrightarrow{AB}$
      2. Point B
      3. plane ABC and plane ABD
      4. Point A
   e. $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$ is
      1. plane ABC
      2. the empty set
      3. plane ABD
      4. none of these.

Chapter 5

FACTORING AND PRIMES

5-1. The Building Blocks of Arithmetic.

It is anticipated that this chapter will take 16-18 days. A less able class may need more time and a more able class may make faster progress; but it is important not to push pupils faster than they can assimilate the material.

There is an opportunity here for the teacher to point out that certain similarities and differences exist in the operations of addition and multiplication of counting numbers. Both have the commutative and associative properties but the distributive property applies only to multiplication over addition. Addition is not distributive over multiplication. The point made in the text is that the counting numbers may be built by using 1 and addition. They cannot be built by using 1 (or any other counting number) and multiplication.

To construct the counting numbers by addition and 1, we start with 1 and add 1 as follows: $1 + 1 = 2$ and $2 + 1 = 3$, and $3 + 1 = 4$, and so on. In multiplication, $1 \cdot 1 = 1$ and this product multiplied by 1 is still 1 so that we cannot build the next counting number. The reason for this is that 1 is the identity element in multiplication. Note that zero is the identity element in addition and we cannot use zero and addition to build the counting numbers. Also, if we use 2 and multiplication, we have $2 \cdot 2 = 4$, and $4 \cdot 2 = 8$, and $8 \cdot 2 = 16$. We thus build some of the numbers but not all of them.

If the example for Exercises 5-1 is not enough, try additional ones as $28 = 7 \cdot 4$ and $38 = 2 \cdot 19$; 13 (no); 29 (no). It may be necessary to emphasize that the factors must each be smaller than the product. This excludes the response: $13 = 1 \cdot 13$.

In these exercises a response may be $12 = 2 \cdot 6$ or $12 = 6 \cdot 2$. The commutative property of addition may again be called to the attention of the pupils. Also, there may be different possible choices of factors as in $12 = 4 \cdot 3$ or $6 \cdot 2$.

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Answers to Exercises 5-1

1. $6 \cdot 2$; $3 \cdot 4$
2. $2 \cdot 18$; $3 \cdot 12$; $4 \cdot 9$; $6 \cdot 6$
3. no
4. no
5. $2 \cdot 4$
6. no
7. $7 \cdot 5$
8. no
9. $3 \cdot 13$
10. $2 \cdot 21$; $3 \cdot 14$; $6 \cdot 7$
11. $2 \cdot 28$; $4 \cdot 14$; $7 \cdot 8$
12. no
13. $2 \cdot 41$
14. $5 \cdot 19$
15. no
16. $2 \cdot 42$; $3 \cdot 28$; $4 \cdot 21$; $6 \cdot 14$; $7 \cdot 12$
17. $3 \cdot 29$
18. no

5-2. Multiples.

If the question arises, a number is a multiple of itself, so that 6 is a multiple of 6 since $6 \cdot 1 = 6$. *Zero is a multiple of every counting number*, since $n \cdot 0 = 0$ where n is any counting number. In the sieve to be made later, the first number in a collection of multiples is circled. These numbers are multiples but they are prime because they are not multiples of *smaller* numbers different from one. All numbers are multiples of one.

Answers to Exercises 5-2

1-9. Any three multiples for each number are acceptable.

10. 1 ② 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9 ~~10~~ 11 ~~12~~ 13 ~~14~~ 15 ~~16~~
17 ~~18~~ 19 ~~20~~

11. 1 2 ③ 4 5 ~~6~~ 7 8 ~~9~~ 10 1 ~~12~~ 13 14 ~~15~~ 16
17 ~~18~~ 19 20

12. 13, 26, 39, 52, 65, 78, 91

13. $\underline{7}$ ⑱ 40 (90) $\underline{84}$ 19 47 $(\underline{63})$ 127 $(\underline{252})$ 25
$\underline{35}$ ㊱ $\underline{273}$ 48 $\underline{105}$

14. 56, 64, 72, 80, 88, 96

15. SSS is a multiple of 111.
111 = 3 · 37
The only two digit multiples of 37 are 37 and 74, hence E must be either 4 or 7.
If E = 4 then consider the products
(14) (74), (24) (74), (34) (74) etc.
However, (14) (74) > 1000, hence, all products ( ) (74) > 1000 and E cannot be 4.
If E = 7, consider the products
(17) (37) = 629
(27) (37) = $\underline{999}$.
W = 2, M = 3, E = 7, S = 9
and the answer is

$$\begin{array}{r} 27 \\ \underline{37} \\ 999 \end{array}$$

Exercises 5-2 may be supplemented and more multiples asked for if the pupils appear to need more practice with multiplication facts. Here is a disguised opportunity for drill on these facts.

5-3. Primes.

In making the "sieve of Eratosthenes", which should be a class exercise, the pupils have the opportunity to discover the set of prime numbers less than 100. At the same time they are learning to use the word "multiple". It will save class time for the teacher to have prepared on ditto sheets the numbers from 1 to 100 arranged in columns of 10. In view of the questions following Problem 15 of Exercises 5-3a, the numbers from 100 to 200 could be included on the sheet. It would be wise to separate the two sets of numbers. The word "sieve" has been used because all the numbers that are not prime numbers, except the number one, are sifted out and prime numbers remain. It should be stressed that the number one is a special number that is neither prime nor a multiple of a prime number. The number two is the only even number that is a prime number. The sifting process is complete after the multiples of 7 are crossed out. The reason for this is that 7 is the largest prime less than the square root of 100.

If the pupils ask for an explanation, the following is suggested. If the sum of two terms is n, it is not possible that both terms are greater than $\frac{n}{2}$. In the same way, if the product of two factors is n, it is not possible that both factors are greater than $\sqrt{n}$. Consequently, if we are searching for factors of a whole number n, and find that there is none which is less than $\sqrt{n}$, then we may be sure that n is prime. Hence, all numbers less than 100 which have no factor less than 10 are prime since for all such numbers n, $\sqrt{n} < 10$.

Possible explanations for not circling 1 in the list of numbers may include the statement that if 1 were circled and all the rest of its multiples were crossed out, no numbers would remain. Actually the decision to exclude 1 from the primes is arbitrary. Some writers include it and later in theorems where it does not apply, (as in unique factorization) specifically speak of all primes except 1.

Numbers 1 to 100

1	②	③	~~4~~	⑤	~~6~~	⑦	~~8~~	~~9~~	~~10~~
⑪	~~12~~	⑬	~~14~~	~~15~~	~~16~~	⑰	~~18~~	⑲	~~20~~
~~21~~	~~22~~	㉓	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	㉙	~~30~~
㉛	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	㊲	~~38~~	~~39~~	~~40~~
㊶	~~42~~	㊸	~~44~~	~~45~~	~~46~~	㊼	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	(53)	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	(59)	~~60~~
(61)	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	(67)	~~68~~	~~69~~	~~70~~
(71)	~~72~~	(73)	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	(79)	~~80~~
~~81~~	~~82~~	(83)	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	(89)	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	(97)	~~98~~	~~99~~	~~100~~

Numbers 101 to 200

$\textcircled{101}$	~~102~~	$\textcircled{103}$	~~104~~	~~105~~	~~106~~	$\textcircled{107}$	~~108~~	$\textcircled{109}$	~~110~~
~~111~~	~~112~~	$\textcircled{113}$	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	~~119~~	~~120~~
~~121~~	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	$\textcircled{127}$	~~128~~	~~129~~	~~130~~
$\textcircled{131}$	~~132~~	~~133~~	~~134~~	~~135~~	~~136~~	$\textcircled{137}$	~~138~~	$\textcircled{139}$	~~140~~
~~141~~	~~142~~	~~143~~	~~144~~	~~145~~	~~146~~	147	~~148~~	$\textcircled{149}$	~~150~~
$\textcircled{151}$	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	$\textcircled{157}$	~~158~~	~~159~~	~~160~~
~~161~~	~~162~~	$\textcircled{163}$	~~164~~	~~165~~	~~166~~	$\textcircled{167}$	~~168~~	~~169~~	~~170~~
~~171~~	~~172~~	$\textcircled{173}$	~~174~~	~~175~~	~~176~~	177	~~178~~	$\textcircled{179}$	~~180~~
$\textcircled{181}$	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	187	~~188~~	~~189~~	~~190~~
$\textcircled{191}$	~~192~~	$\textcircled{193}$	~~194~~	~~195~~	~~196~~	$\textcircled{197}$	~~198~~	$\textcircled{199}$	~~200~~

Answers to Exercises 5-3a

(It is suggested that at least the first 6 of these exercises should be answered in class, with discussion.)

1. The multiples of 11 were all crossed out in advance except for 11 itself.
2. The multiples of 7 finished the crossing out process.
3. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.
4. 25.
5. 15.
6. All numbers would be crossed out except 1.
7. 10, 20, 30, 40, 50, 60, 70, 80, 90, 100.
8. 7, 14, 21, 28, 35, 42, 49.
9. 33, 36, 39, 42, 45, 48, 51, 54, 57.
10. 15, 30, 45, 60, 75, 90.
11. a. 8
    b. 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43, 59 and 61, 71 and 73.
12. Yes. 3, 5, 7.
13. a, b, c.

$\underline{1}$ $\textcircled{2}$ $\textcircled{\underline{3}}$ 4 $\textcircled{\underline{5}}$ 6 $\textcircled{\underline{7}}$ 8 $\underline{9}$ 10 $\textcircled{\underline{11}}$ 12
$\textcircled{\underline{13}}$ 14 $\underline{15}$ 16 $\textcircled{\underline{17}}$ 18 $\textcircled{\underline{19}}$ 20.

d. Yes. 2.

*14. 3, 5, 7, 11, 13, 17, 19, 23, 29.

*15. a. 8 is a multiple of 2.

b. $7 \cdot 7$ is 49, but $11 \cdot 11$ is 121. (The largest prime less than $\sqrt{100}$ is 7. See explanation under 5-3 Primes.)

The questions following Problem 15 of Exercises 5-3a are still more difficult and are not intended for all pupils. Some pupils, however, may wish to continue the sieve for numbers from 101 to 200. Such work should be entirely voluntary.

*16. There are 21 prime numbers in the set of numbers from 101 to 200. The primes are 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199.

*17. Twin primes are 101 and 103; 107 and 109; 137 and 139; 149 and 151; 179 and 181; 191 and 193; 197 and 199.

*18. There are no prime triplets except 3, 5, 7 because every third odd number after 3 is a multiple of 3.

If your pupils are curious about primes, they may be interested in some of the unanswered questions which have been raised. Primes seem to increase in an irregular and mysterious way. Mathematicians have searched for years for laws about the distribution of primes. In his book on geometry, Euclid proved that the series of primes never comes to an end. No matter how big a counting number you choose, there is a prime larger than that number.

Here is the simplest proof that there is no largest prime number.

Let p be a prime number. Let N be the product of all prime numbers from 2 to p.

$$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \ldots \cdot p.$$

Look at $N + 1$.

$$N + 1 = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \ldots \cdot p) + 1.$$

This number leaves a remainder of 1 when divided by any of the prime numbers 2, 3, 5, 7, 11, 13, ..., p. Hence $N + 1$ is not divisible by any of these primes. If $N + 1$ can be factored into prime factors then each of its prime factors is greater than p. On the other hand if $N + 1$ is a prime, then it is a prime greater than p. Thus, we have shown a method for constructing a prime greater than any given prime.

Examples are:

$$2 \cdot 3 + 1 = 7 \qquad \text{prime}$$

$$2 \cdot 3 \cdot 5 + 1 = 31 \qquad \text{prime}$$

$$2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211 \qquad \text{prime}$$

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = 2311 \qquad \text{prime}$$

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 \qquad = 59 \cdot 509$$

Euclid proved that there are in the series of primes as large gaps as you please. For example, let "N" be an abbreviation for the number:

$$N = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ \ldots \ \cdot 101,$$

obtained by multiplying all the counting numbers from 1 to 101. In ordinary decimal notation, N is a number of about 150 digits. Then none of the numbers

$$N + 2, N + 3, \ldots, N + 101$$

is a prime. For example $N + 2$ is a multiple of 2 since

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot 101 + 2 = 2(1 \cdot 3 \cdot 4 \cdot \ldots \cdot 101 + 1).$$

$N + 3$ is a multiple of 3, etc. Here are 100 consecutive counting numbers which are not primes. So we have found a gap of at least 100 in the series of primes.

In his investigation of the large gaps in the series of primes, the Russian mathematician, Chebyshev, proved that between any counting number and its double, there is at least one prime. In mathematical language, if n is any counting number, then there is a prime p for which

$$n \leq p \leq 2 \cdot n$$

This is a way of stating that there cannot be too large a gap between any prime and the next prime.

It is unknown whether there is a prime between any two consecutive squares. As far as anyone has calculated, in the sequence of squares:

$$1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 9, \quad 16, \quad 25, \quad 36, \ldots$$

we have always found at least one prime between any two consecutive squares. No one has ever proved that this is always true, and no one has found two squares between which there is no prime.

The small gaps in the series of primes are also mysterious. After the prime, 3, the smallest possible gap is 2.

On studying tables of primes, we find that the twin primes become increasingly rare. The unsolved problem is: Do the twin primes become so rare that beyond a certain number there are no more twin primes? The other possibility is that no matter how large a number we choose, we can always find twin primes larger than this number.

The primes are the building blocks of the system of counting numbers when we use the operation of multiplication, and the idea of "prime number" comes from the operation of multiplication. Some of the most fascinating problems arise when we try to relate primes to the operation of addition.

In 1742 (?) the mathematician, Goldbach, wrote to the great Swiss mathematician, Euler. He had noticed that every even number from 4 on seems to be the sum of two primes.

For example,

$4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$ $10 = 5 + 5$

$20 = 7 + 13$, $32 = 13 + 19$, $64 = 3 + 61$, etc.

Goldbach wrote that he had not been able to find an even number greater than 2 which is not the sum of two primes. He asked Euler whether it is true that every even number greater than 2 is the sum of two primes. Neither Euler, not any other mathematician up to now, has been able to prove or disprove this statement.

Answers to Exercises 5-3b

1. 90, 91, 92, 93, 94, 95, 96.

2.

Number	Twin Primes Contained in This Series	Number of Twin Primes
a. 1 - 16	(3,5), (5,7), (11,13)	3
17 - 32	(17, 19), (29, 31)	2
33 - 48	(41, 43)	1
49 - 64	(59, 61)	1

2. (continued)

Number	Twin Primes Contained in this Series	Number of Twin Primes
65 - 80	(71, 73)	1
81 - 96	none	0
97 - 112	(101, 103), (107, 109)	2
113 - 128	none	0

b. Since there are series of numbers where there are no primes we might think primes occur less often as the series become larger.

3. A difference of 1 means that one of the primes must be even. The only even prime is 2.
4. All other even numbers are multiples of 2.
5. (5,11), (7,13), (11,17), (13,19), (17,23), (23,29), (31,37), (37,43), (41,47), (47,53), (53,59), (67,73), (73,79), etc.
6. (89, 97) The largest difference is 8.
7. a. 5 + 7
   b. 7 + 7, 11 + 3
   c. 13 + 3, 11 + 5
   d. 3 + 47, 7 + 43, 13 + 37, 19 + 31
   e. 73 + 3, 5 + 71, 17 + 59, 23 + 53, 47 + 29
   f. 11 + 83, 89 + 5, 23 + 71
8. 181. The sequence consists of the prime numbers that end with the digit 1.

5-4. Factors.

The purposes of this section are to develop understandings of the terms "factor," "complete factorization," and "composite number." The unique factorization property is discussed, and the property should be understood even if its name is difficult. The pupils may prefer to say that no matter how the factors of a number are found, the same set of factors always occurs in the complete factorization of a number. The order may differ but the factors will be the same.

As a matter of interest, the unique factorization property is usually referred to as the Fundamental Theorem of Arithmetic.

Notice that the definition of factor includes zero. Also, zero has factors (of which one must always be zero). Problem 4 of Exercises 5-4a serves as a basis of discussion of zero. So does Problem 5 of Exercises 5-4c.

In the explanation leading to the definition of a composite number the expression "2 distinct factors" is often used. For our readers we choose to use the word "different" to emphasize the requirement that the factors are not alike.

As the pupils write the factors in different order, opportunity arises to show how the commutative and associative properties of multiplication are used to re-arrange the factors. In the complete factorization for a prime number like 17, only one factor is written. We do not write $17 \cdot 1$ since 1 is not a prime number.

It is not mandatory that pupils use exponents. They should be encouraged to use them, however, wherever it is reasonable to do so.

Answers to Exercises 5-4a

1. a. $5 \cdot 3$
   b. $7 \cdot 2$
   c. $11 \cdot 3$
   d. $16 \cdot 2$ or $8 \cdot 4$
   e. $21 \cdot 2$ or $7 \cdot 6$ or $3 \cdot 14$
   f. $9 \cdot 6$ or $27 \cdot 2$ or $3 \cdot 18$
   g. $13 \cdot 1$
   h. $1 \cdot 1$
   i. $0 \cdot$ any number
2. a. $\{1, 2, 5, 10\}$
   b. $\{1, 31\}$
   c. $\{1, 2, 3, 6, 9, 18\}$
   d. $\{1, 3, 7, 21\}$
   e. $\{1, 7, 11, 77\}$
   f. $\{1, 2, 3, 5, 6, 10, 15, 30\}$
   g. $\{1, 13\}$
   h. $\{1,3,9\}$

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3. 13, 31
4. zero
5. a. 1, 2, 5, 10
   b. 3
   c. 1
   d. 2, 3, 6
6. a. 5
   b. 8
   c. 30
   d. 0
   e. 1
   f. any number
   g. 2
   h. 2
   i. 13
   j. 17
   k. 18
   l. 2

Answers to Oral Exercises 5-4b

1. a. 2 has factors 1, 2
   b. 9 has factors 1, 3, 9
   c. 26 has factors 1, 2, 13, 26
   d. 61 has factors 1, 61
   e. 133 has factors 1, 7, 19, 133
   f. 97 has factors 1, 97
   g. 52 has factors 1, 2, 4, 13, 26, 52
   h. 79 has factors 1, 79.
   9, 26, 133, and 52 have more than two different factors.
2. $9 = 3 \cdot 3$
   $26 = 2 \cdot 13$
   $133 = 7 \cdot 19$
   $52 = 2 \cdot 26$ or $4 \cdot 13$
3. a. $2 \cdot 3$ b. $3 \cdot 5$ c. $2 \cdot 2 \cdot 2 \cdot 2$ d. $2 \cdot 2 \cdot 3$
   e. 31 f. $5 \cdot 7$

Answers to Exercises 5-4c

1. a. Complete
   b. Not complete
   c. Not complete
   d. Complete
   e. Complete
   f. Not complete
2. a. Composite
   b. Prime
   c. Composite
   d. Composite
   e. Composite
   f. Composite
   g. Prime

3. a. $5 \cdot 2$ c. 3^2
d. $3^2 \cdot 2$ e. 3^3
f. $3 \cdot 2^3$

4. a. $26 \cdot 1$, $13 \cdot 2$
b. $38 \cdot 1$, $19 \cdot 2$
c. $36 \cdot 1$, $18 \cdot 2$, $12 \cdot 3$, $9 \cdot 4$, $6 \cdot 6$
d. $68 \cdot 1$, $34 \cdot 2$, $17 \cdot 4$
e. $81 \cdot 1$, $27 \cdot 3$, $9 \cdot 9$
f. $100 \cdot 1$, $50 \cdot 2$, $25 \cdot 4$, $20 \cdot 5$, $10 \cdot 10$

5. a. No, zero is not a factor of 6 since there is not a number which, when multiplied by zero, gives a product of 6.
b. Yes, 6 is a factor of zero since the product of 6 and zero is zero. Thus, the definition is satisfied.

6. a. $\{1, 2, 4, 5, 10, 20\}$
b. $20 = 2 \cdot 2 \cdot 5 = 2^2 \cdot 5$
c. 1, 4, 10, 20.

7. a. $3 \cdot 5^2$ d. $3^2 \cdot 5$
b. 2^6 e. $2^3 \cdot 7$
c. $3 \cdot 5 \cdot 7$ f. $2 \cdot 5^2$

5-5. Divisibility.

Divisibility is a property of a number. It is the number, not the numeral, which is divisible by another number. The numeral is a way of writing the number. In base ten, a numeral which represents an even number ends with an even number. In base five this is not necessarily so. This is illustrated in Problem 5 of Exercises 5-5a.

The idea that $24 = 8 \cdot a$ also means $24 \div 8 = a$ is presented here to begin the preparation of pupils for Chapter 6. It is not necessary that it be stressed heavily at this time. Casual treatment may be better. Each time it is touched upon the meaning should become clearer.

Answers to Exercises 5-5a

1. No.
2. Even, since 0 is divisible by 2.
3. a. even
   b. even
   c. even
   d. even
   e. odd
   f. even
   g. odd
   h. odd
4. a. even
   b. even
   c. odd
   d. even
   e. even
   f. odd
   g. even
   h. odd
   i. odd
   j. odd
   k. even
   l. even
5. a. even
   b. odd
   c. even
   d. odd
   e. even
   f. even
6. a. even
   b. odd
   c. even
   d. even
7. Divisibility is a property of a number.
   See the discussion on the previous page.
8. Even. The number $2 \cdot a$ (where a is a whole number) is divisible by 2 since 2 is a factor.

In the development of tests for divisibility, it may be necessary to use additional numerical examples for slower pupils.

In the text we do not give the test for divisibility by 3 as we hope the student will be able to discover it himself. The test he should eventually find is:

A number is divisible by 3 if the sum of the digits in its decimal numeral is a number which is divisible by 3.

In the exercises the pupils are asked to develop tests for divisibility by 10, 5, 9, 4 and 6.

It should be made clear that a demonstration which shows that a rule works for several examples does not constitute a proof that it is true for all cases. Only one counter-example is needed to show that a statement is not true in general.

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Tests for divisibility for 7, 11, 13, for example, exist. See Mathematics for Junior High School, Supplementary Units (SMSG): Unit 3.

Answers to Exercises 5-5b

1. a. yes
   b. no
   c. no
   d. yes
   e. yes
   f. yes

2. a. 17
   b. 30 R 2
   c. 28 R 2
   d. 71
   e. 3334
   f. 484

3. 10, 20, 30, 40, 50, 60, 70, 80, 90, 100.
   Each numeral has 0 in the ones place.

4. 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80.
   Each numeral has 0 or 5 in the ones place.

5. Typical examples may be:

$$9 \times 62 = 558, \quad 5 + 5 + 8 = 18$$
$$9 \times 568 = 5112, \quad 5 + 1 + 1 + 2 = 9$$

6. $1\underline{28} \div 4 = 32;\ \underline{28} = 7 \cdot 4$
   $4\underline{13} \div 4 = 103$ R 1; $\underline{13}$ is not divisible by 4
   $50\underline{12} \div 4 = 1253;\ \underline{12} = 3 \cdot 4$
   $1\underline{09} \div 4 = 27$ R 1; $\underline{09}$ is not divisible by 4

7. a. [✓ 2] 3 [✓ 4] 5 9 10
   b. [✓ 2] [✓ 3] 4 [✓ 5] 9 [✓ 10]
   c. [✓ 2] 3 [✓ 4] 5 9 10
   d. 2 [✓ 3] 4 [✓ 5] 9 10
   e. 2 [✓ 3] 4 5 [✓ 9] 10
   f. [✓ 2] [✓ 3] [✓ 4] 5 9 10
   g. [✓ 2] [✓ 3] [✓ 4] 5 [✓ 9] 10
   h. 2 [✓ 3] 4 5 [✓ 9] 10
   i. [✓ 2] [✓ 3] [✓ 4] [✓ 5] [✓ 9] [✓ 10]

j. [✓ 2] [✓ 3] [✓ 4] 5 [✓ 9] 10
k. 2 3 4 5 9 10
l. [✓ 2] 3 4 [✓ 5] 9 [✓ 10]

8. Yes, because 3 is a factor of 9.

9. a. Yes, because $2 \cdot 3$ is 6.
b. Yes, because 2 and 3 are factors of 6.

10. a. 144 b. 102 c. No. d. 504

Answers to Exercises 5-5c

1. 5
2. 2
3. 7
4. 2
5. 3
6. 2
7. 3
8. 11
9. $2 \cdot 23$
10. $5 \cdot 13$
11. $2 \cdot 3 \cdot 19$
12. $2 \cdot 7^2$
13. $2^2 \cdot 3^2 \cdot 5$
14. $2 \cdot 3 \cdot 43$
15. $2 \cdot 3^5$
16. $2^2 \cdot 3 \cdot 37$
17. $5^3 \cdot 3$
18. $7^2 \cdot 11$
19. 3^2
20. $2^2 \cdot 5^2$
21. 2^6
22. 37
23. $2^5 \cdot 5^5$
24. 11
25. $2^3 \cdot 5^3$
26. 2^5

*27. 3^4, $2^2 \cdot 5^4$, 2^{12}, 37^2, $2^{10} \cdot 5^{10}$, 11^2, $2^6 \cdot 5^6$, 2^{10}

*28. An even number of times. No.

*29. It is always the square of a prime: 4: {1, 2, 4}, 9: {1, 3, 9}, 25: {1, 5, 25}, etc.

*30. 1, 2, 3; sum = 6 1, 2, 4, 7, 14; sum = 28
Typical example: 15: 1, 3, 5; sum = 9
8: 1, 2, 4; sum = 7

Thus a perfect number is equal to the sum of all its factors less than itself.

Note that if we find the complete factorization of a number that is a power of 10, the prime factors 2 and 5 appear to the same powers in the factorization.

That is: $10 = 2 \cdot 5$; $100 = 10^2 = 2^2 \cdot 5^2$
$1000 = 10^3 = 2^3 \cdot 5^3$, etc.

5-6. Least Common Multiple.

The purpose of this section is to develop skill and understanding in finding common multiples of several numbers and in finding the least common multiple of several numbers. Pupils are given the method of listing multiples of each in the text. The method of using the complete factorization of each number is discussed in Problem 11. For very slow pupils the second way may be too difficult.

It is recommended that the phrase, least common multiple, instead of the abbreviation LCM be used in oral work. Stress that "multiple" in this sense means "multiple of a counting number."

Zero is not acceptable as a multiple here. If zero were allowed as a multiplier, then every number would have zero as its least multiple since $0 \cdot n = 0$ for any number n.

Common multiples are investigated before the least common multiple is introduced. The use of the LCM in work with fractions again provides opportunity for drill in fundamentals. Exercises 5-6 use multiplication facts and here pupils may be asked to work with multiples which involve multiplication facts they are uncertain of as they list multiples of 7, 8, and 9.

Notice that only the denominators of fractions are discussed. The problem of writing different forms or changing $\frac{3}{4}$ to $\frac{9}{12}$ is treated in Chapter 6 and could not be developed here.

The method of using complete factorization is the direct method. It may be too difficult for slower pupils. For teachers who wish to develop it, the following examples are included.
Examples: Find the least common multiple of 12 and 18.

$$12 = 2^2 \times 3 \qquad\qquad 18 = 2 \times 3^2$$

Any number which is a multiple of 12 must have 2^2 as a factor and also 3 as a factor.

Any number which is a multiple of 18 must have 2 as a factor and also 3^2 as a factor.

Thus a number which is a common multiple of 12 and 18 must have among its factors all of the following:

$$2^2, 3, 2, 3^2$$

But a number which has 2^2 as a factor certainly has 2 as a factor. Similarly, a number which has 3^2 as a factor certainly has 3 as a factor. Thus any number which has the two factors 2^2 and 3^2 will have all four of the factors $2^2, 3, 2, 3^2$. The smallest number which has the factors 2^2, and 3^2 is $2^2 \times 3^2 = 36$.

On the other hand, $2^2 \times 3^2$ is a multiple of 12 (which is $2^2 \times 3$) and it is also a multiple of 18 (which is 2×3^2). It is their least common multiple. Notice that the power of each prime factor in the least common multiple is the larger of the powers to which it occurs in the two given numbers.

Summary of Method: (1) Find the complete factorization of each number. (2) Notice which primes occur in at least one of the factorizations. (3) For each prime noticed in (2), write the largest power of it which occurs in any of the factorizations. (4) Multiply all the powers written in (3); this product is the least common multiple.

Answers to Exercises 5-6

1. a. The set of multiples of 6 less than 100:
   $\{6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96\}$
   b. The set of multiples of 8 less than 100:
   $\{8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96\}$
   c. The set of multiples of 9 less than 100:
   $\{9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99\}$
   d. The set of multiples of 12 less than 100:
   $\{12, 24, 36, 48, 60, 72, 84, 96\}$
2. a. The set of common multiples of 6 and 8 less than 100:
   $\{24, 48, 72, 96\}$
   b. The set of common multiples of 6 and 12 less than 100:
   $\{12, 24, 36, 48, 60, 72, 84, 96\}$

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c. The set of common multiples of 8 and 12 less than 100: {24, 48, 72, 96}

d. The set of common multiples of 8 and 9 less than 100: {72}

3. a. The least common multiple of 6 and 8 is 24.
   b. The least common multiple of 6 and 12 is 12.
   c. The least common multiple of 8 and 12 is 24.
   d. The least common multiple of 8 and 9 is 72.

4. a. 10 c. 30 e. 30 g. 42
   b. 12 d. 12 f. 60 h. 72

5. a. 6
   b. 15
   c. 21
   d. 35
   e. 22
   f. 26
   g. 77
   h. 39
   i. 143
   j. 30

6. The least common multiple of 3 numbers is the smallest counting number which is a multiple of each of them.

7. The least common multiple of any set of numbers is the smallest counting number which is a multiple of each number of the set of numbers.

8. a. 6 b. 6

9. The least common multiple of a set of numbers is at least as large as the largest number of the set.

10. a. 12 c. 20 e. 40 g. 60 i. 30
    b. 8 d. 18 f. 60 h. 60 j. 24

*11. a. 48 c. 45 e. 112 g. 72 i. 660
    b. 144 d. 70 f. 60 h. 360 j. 720

*12. a. No. b. No.
    c. No. Any multiple larger than a given multiple can be found.

*13. a. The product of 2 counting numbers is the LCM of the 2 numbers when they have no common factors except 1.
    b. -- when they have no common factors except 1.

*14. He placed 1 in the first cup, 1 in the second and 18 in the third, and said that all were odd since 18 is an odd number of lumps of sugar in a cup of coffee.

5-8. Chapter Review.

Answers to Exercises 5-8

1. 2, 3, 5, 7, 11

2. {41, 43, 47, 53, 59}

3. Yes, 2

4. a. $8 = 5 + 3$
   b. $2^4 = 11 + 13 = 19 + 5 = 17 + 7$

5. 2, 3; No, any other set of two consecutive numbers contains a number divisible by two.

6. {10, 20, 30, 40, 50}

7. $2^4 \cdot 3^2$

8. a. Composite
   b. Prime
   c. Composite
   d. Prime
   e. Composite
   f. Composite

9. a. {1, 2, 4, 8}
   b. {1, 2, 3, 4, 6, 12}
   c. {1, 2, 4, 8, 16}
   d. {1, 19}
   e. {1, 3, 7, 21}
   f. {1, 2, 4, 7, 14, 28}

10. a. {1, 2, 4}
    b. {1, 3}
    c. {1}
    d. {1, 2, 4}
    e. {1, 2, 4}
    f. {1}

11. a. $5 \cdot 7$
    b. $2 \cdot 2$ or 2^2
    c. $3 \cdot 3 \cdot 3$ or 3^3
    d. $2 \cdot 3 \cdot 11$
    e. $2 \cdot 2 \cdot 3 \cdot 5$ or
    $2^2 \cdot 3 \cdot 5$
    f. $3 \cdot 7 \cdot 7$ or $3 \cdot 7^2$

12. a. $2 \cdot 3 \cdot 13$
    b. $2 \cdot 2 \cdot 2 \cdot 7$ or $2^3 \cdot 7$
    c. $2 \cdot 3 \cdot 3$ or $2 \cdot 3^2$
    d. $2 \cdot 5 \cdot 5 \cdot 5$ or $2 \cdot 5^3$
    e. $2 \cdot 2 \cdot 2 \cdot 2 \quad 2 \cdot 3$ or $2^5 \cdot 3$
    f. $2 \cdot 5 \cdot 11$

13. a. Yes, 1; Yes, 1.
    b. Yes, 3.

14. a. 100 b. 42 c. 44

15. b is a whole number, a multiple of 2, a multiple of c, an even number, divisible by 2, divisible by c, and has the factors 2 and c.

16. a. 2 [✓ 3] 4 [✓ 5] 6 9 10
b. [✓ 2] [✓ 3] 4 5 [✓ 6] 9 10
c. [✓ 2] [✓ 3] [✓ 4] [✓ 5] [✓ 6] 9 [✓ 10]
d. 2 3 4 [✓ 5] 6 9 10
e. 2 [✓ 3] 4 5 6 [✓ 9] 10

17. a. 36 d. 40
b. 30 e. 24
c. 16 f. 120

18. a. 2 d. 12
b. 7 e. 2
c. 11 f. 5

19. ---

20. Always.

21. Sometimes.

22. After 15 minutes they will strike together.

5-9. Cumulative Review.

Answers to Exercises 5-9

1. 20

2. a. Base 4 b. 32_{four}

3. 37_{eight}

4. a. 21 b. 21 c. 31

5. b. odd

6. $(2 \times 10^3) + (3 \times 10^2) + (0 \times 10) + (4 \times 1)$

7. a. Division. b. Subtraction.

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8. a. Yes. b. No.

9. a. 8
   b. Any whole number.
   c. 0

10. The associative property of multiplication says that if we multiply three numbers, we may group them any way we please without changing the final result.

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

11. a. 8 is less than 12.
   b. 3^4 is greater than 32.
   c. 5 is larger than 3 and 3 is larger than 2.

12. a. $\overleftrightarrow{CD}$
   b. The empty set.
   c. Many examples such as $\overleftrightarrow{AB}$ and $\overleftrightarrow{CJ}$, $\overleftrightarrow{GH}$ and $\overleftrightarrow{DK}$, etc.
   d. Point D.
   e. Point C.

Sample Test Questions for Chapter 5

Teachers should construct their own tests, using carefully selected items from those given here and from their own. There are too many questions here for one test. Careful attention should be given to difficulty of items and time required to complete the test.

I. True-False Questions

(T) 1. Every composite number can be factored into prime numbers in exactly one way, except for order.

(F) 2. The sum of an odd and an even number is always an even number.

(T) 3. Some odd numbers are not primes.

(F) 4. Every composite number has only two prime factors.

(F) 5. The number 51 is a prime.

(T) 6. All even numbers have the factor 2.

(F) 7. Any multiple of a prime number is a prime.

(T) 8. 8 can be expressed as the sum of twin primes.

(T) 9. Even though 1 has as factors only itself and 1, it is not considered a prime number.

(F) 10. All odd numbers have the factor 3.

(F) 11. No even number is a prime.

(T) 12. The least common multiple of 3, 4, and 12 is 12.

(F) 13. The number one is a prime factor of all counting numbers.

(F) 14. The greatest common multiple of 2, 5 and 10 is 100.

(T) 15. The difference between any two prime numbers greater than 100 is always an even number.

(F) 16. The number 41 is a composite number.

(F) 17. A multiple of 6 must be a multiple of 18.

(F) 18. The least common multiple of 2 and 6 is 12.

(T) 19. The greatest common factor of any two even numbers is at least 2.

II. Multiple Choice

20. Which of the following is an odd number?

(a) 21_{five}

(b) 33_{seven}

(c) 10_{two}

(d) 18_{twelve}

(e) None of the above 20 (a)

21. The greatest common factor of 48 and 60 is:

(a) 2 × 3

(b) 2 × 2 × 3

(c) 2 × 2 × 2 × 2 × 3 × 5
(d) 2 × 2 × 2 × 2 × 2 × 2 × 3 × 3 × 5
(e) None of the above. 21 (b)

22. Every counting number has at least the following factors:
(a) Zero and one
(b) Zero and itself
(c) One and itself
(d) Itself and two
(e) None of the above 22 (c)

23. In the complete factorization of a number
(a) All the factors are primes.
(b) All the factors are composites.
(c) All the factors are composite except for the factor 1.
(d) All the factors are prime except for the factor 1.
(e) None of the above 23 (a)

24. How many different prime factors does the number 72 have?
(a) 0
(b) 1
(c) 2
(d) 3
(e) None of the above 24 (c)

25. The least common multiple of 8, 12, and 20 is:
(a) 2 × 2
(b) 2 × 3 × 5
(c) 2 × 2 × 2 × 3 × 5
(d) 2 × 2 × 2 × 2 × 2 × 2 × 3 × 5
(e) None of the above 25 (c)

26. Which of the following is an even number?
(a) $(100)_{\text{three}}$
(b) $(100)_{\text{five}}$
(c) $(100)_{\text{seven}}$
(d) $(100)_{\text{twelve}}$
(e) None of the above 26 (d)

27. Which of the following numbers is odd?
(a) 17×18
(b) 18×11
(c) 11×20
(d) 99×77
(e) None of the above. 27 (d)

28. Which of the following is not a prime number? There is only one.
(a) 271 (d) 282
(b) 277 (e) 283
(c) 281 28 (d)

29. Let a represent an odd number, and b represent an even number; then $a + b$ must represent
(a) an even number. (d) a composite number.
(b) a prime number. (d) None of these.
(c) an odd number. 29 (c)

30. If n represents an odd number, the next odd number can be represented by
(a) $n + 1$ (d) $2 \times n$
(b) $n + 2$ (e) None of these
(c) $n + 3$ 30 (b)

31. A counting number is an even number if it has the factor:
(a) 5 (d) 1
(b) 3 (e) None of these.
(c) 2 31 (c)

32. Which of the following sets contains only even numbers?
(a) $\{2, 3, 4, 5, 9, 10\}$ (d) $\{2, 4, 10\}$
(b) $\{2, 5, 10\}$ (e) $\{3, 9\}$
(c) $\{3, 5, 9\}$ 32 (d)

33. Which of the following is a prime number?
(a) 4 (d) 33
(b) 7 (e) None of these.
(c) 9 33 (b)

34. Which of the following is not a factor of 24?
(a) 2
(b) 3
(c) 4
(d) 9
(e) 12

34 (d)

35. Which of the following is the complete factorization of 36?
(a) 4 × 9
(b) 2 × 3 × 6
(c) 3 × 12
(d) 2 × 18
(e) 2 × 2 × 3 × 3

35 (e)

36. The numbers 8, 9, 16, 20, 27, and 72 are all
(a) prime numbers.
(b) even numbers.
(c) odd numbers.
(d) composite numbers.
(e) none of these.

36 (d)

37. How many multiples of 4 are there between 25 and 50?
(a) 5
(b) 7
(c) 9
(d) 11
(e) none of these

37 (e)

38. If a whole number has 6 as a factor, then it also has the following factors:
(a) 2 and 3
(b) 2 + 3
(c) 12
(d) all multiples of 6
(e) none of these

38 (a)

39. Suppose p and q are counting numbers and q is a factor of p; then:
(a) q is a multiple of p.
(b) p is a multiple of q.
(c) q must be a prime number.
(d) the greatest common factor of p and q must be less than q.
(e) none of these.

39 (b)

40. The greatest common factor of 60 and 42 is
(a) 2 × 3
(b) 2 × 2 × 3
(c) 2 × 3 × 5
(d) 2 × 3 × 7
(e) none of these

40 (a)

41. The product of two factors must be
(a) a composite number.
(b) a prime number.
(c) smaller than one of the numbers.
(d) smaller than both of the numbers.
(e) none of these. 41 (a)

42. The least common multiple of two numbers is always:
(a) their product.
(b) the product of their factors.
(c) the sum of their factors.
(d) the sum of the numbers.
(e) none of these 42 (e)

43. Which of the following statements describes a prime number?
(a) a number which is a factor of a counting number?
(b) a number which has no factors.
(c) a number which does not have 2 as a factor.
(d) a number which has exactly 2 different factors.
(e) none of these. 43 (d)

44. How many prime numbers are there between 20 and 40?
(a) 4 (d) 9
(b) 5 (e) none of these.
(c) 8 44 (a)

45. When two prime numbers are added, the sum is
(a) always an odd number.
(b) always an even number.
(c) always a composite number.
(d) always a prime number.
(e) none of these. 45 (e)

46. The set of factors of the number 12 is
(a) {1, 2, 3, 4, 8, 12} (d) {2, 3, 4, 6, 12}
(b) {1, 2, 3, 4, 6, 12} (e) none of these.
(c) {1, 2, 3, 4, 6} 46 (b)

47. How many different factorizations of two factors each does 75 have?

(a) 2
(b) 3
(c) 4
(d) 5
(e) none of these.

47 (b)

48. The number of factors in the complete factorization of 82 is

(a) 2
(b) 3
(c) 4
(d) 5
(e) none of these.

48 (a)

III. Problems

49. Find the complete factorization of each number.

(a) 16 2^4
(b) 100 $2^2 \times 5^2$
(c) 57 3×19

50. Find the greatest common factor of each set of numbers.

(a) 5 and 25 5
(b) 18 and 27 9
(c) 60, 36 and 24 12

51. Find the least common multiple of each set of numbers.

(a) 6 and 8 24
(b) 7 and 9 63
(c) 16, 12 and 6 48

52. Find the smallest number which has a factorization composed of 3 composite numbers.

$64 = 4^3$.

53. Show that a product is even if one (or more) of its factors is even. At least one factor is 2.

54. Is the set of even numbers closed under addition? Show that your answer is correct. Yes.

$2n + 2m = 2(n + m)$ which is an even number.

55. If the complete factorization of a number is $2 \times 3 \times 5$, what factors less than 20 does the number have?

1, 2, 3, 5, 6, 10, 15.

56. **Find all the common multiples less than 100 of these three numbers:** 3, 6, 9.

18, 36, 54, 72, 90.

57. **Write all the factorizations of two factors for the number 50.**

1 × 50, 2 × 25, 5 × 10.

Chapter 6

RATIONAL NUMBERS AND FRACTIONS

By the time students have reached the seventh grade they have acquired reasonable facility in performing the fundamental operations with fractions.

The rational numbers are presented formally during the latter stages of the primary grades. At this time, the justification for fundamental operations with rational numbers is based upon a series of rationalizations which make use of the number line and subdivided regions in accordance with the maturity level of the students.

A significant and recurring theme in the seventh and eighth years is the structure of the real number system. On this level, it is believed that the student is ready for a more formal and comprehensive understanding of the real number system in terms of the field axioms. The treatment cannot be totally abstract, but it should be mathematically sound. The abstract framework can be indicated by a wealth of concrete illustrations and a well-planned teaching sequence. For example, the properties of the operations with whole numbers such as the commutative, associative, and distributive laws, closure, and identity elements can be illustrated and applied in specific cases.

The purpose of this chapter is to provide for the extension of the number system from the counting numbers to the rational numbers. At this stage, however, the point of view becomes more abstract than before. The need for the invention of the rationals is presented in terms of the failure of closure of the counting numbers under division. After the need for the creation of a system of rational numbers is established, the student is led to conclude that rules for operation with these numbers are necessary if these numbers are to become a functioning part of the number system. It is assumed that these rules are, for the most part, known by the students. In this chapter, the abstract theory upon which these rules rest is carefully developed. The student learns, for example, that the rule for multiplying fractions is not an arbitrary rule, but is, in fact the only rule which is consistent with the preservation of the properties of multiplication.

The premise on which this approach is based is that, having once seen that there are reasons for operating with fractions as we do, the student will perform these operations with greater confidence and better understanding even after the exact nature of the reasons has been forgotten. It is hoped that, having been exposed to this treatment, the student will feel that mathematics has a logical foundation and is not just a collection of rules. This feeling can be established by the mere fact that the explanations are true even though some students may barely understand them. It is believed that this pedagogical principle has as much validity for the less gifted academically as for those with considerable native ability in mathematics.

The outline below is a summary of the way in which these ideas are developed.

SECTION 1 - The student is oriented to the approach that the purpose of the chapter is to understand why the rules of operation with fractions work as they do.

It is shown that the set of counting numbers is not closed under division.

SECTION 2 - The rational numbers are invented.

SECTION 3 - The distinction between rational numbers and their names, fractions, is made.

SECTION 4 - The meaning of division as the inverse of multiplication is explained.

SECTION 5 - A more usable form of the definition of rational number is given.

SECTION 6 - It is explained that if the operations of multiplication and addition are to be extended to operate on the rational numbers then this extension must be made in such a way as to preserve the characteristic properties of these operations. The characteristic properties of multiplication and addition are reviewed.

SECTION 7 - The reason why the multiplication of rational numbers must follow the standard pattern is shown. The special case of the multiplication of two rational numbers where one of these numbers is a whole number is included.

SECTION 8 - The Multiplication Property of 1 is used as the basis for changing the form of a fraction to an "equivalent fraction".

SECTION 9 - The reciprocal is introduced and defined.

SECTION 10 - It is shown that, since division is the inverse operation of multiplication, the division process may be treated from the point of view of multiplication. The standard pattern of multiplying by the reciprocal of the divisor is developed.

SECTION 11 - It is shown that the retention of the distributive property makes it necessary to add rational numbers as we do. The addition of rational numbers with the same denominators and then with different denominators is treated.

SECTION 12 - Subtraction of rational numbers is treated as the inverse operation of addition.

The points at which the greatest difficulties may be expected are Sections 4, 5, and 6. In Sections 4 and 5 it is explained that

$$b \cdot x = a$$

means the same thing as

$$x = a \div b$$

This idea is very important in this development of the rational numbers and is stressed in many problems both before and after this section. These problems may seem quite repetitious but, for many students, a great many problems will be needed to drive home the idea so that when it is used in a rather complicated setting in Section 8 the student will be on familiar ground.

In Section 6 the definition of $\frac{a}{b}$ as the number x for which $b \cdot x = a$, and the pattern $b \cdot \frac{a}{b} = a$, are developed. The succeeding treatment is based upon this definition.

In Section 6 the need for preserving the properties of the arithmetical operations when they are extended to operate on the rational numbers is supported by an analogy. It is hoped that the students will find this analogy convincing. The idea is fundamental to the rest of the chapter.

In Section 7 it is shown how the formula

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

is an inescapable consequence of the properties of multiplication. The argument has been simplified as much as possible and an attempt has been made to develop it in a series of questions to which answers are supplied in the text. Some students will be able to follow this development only in a general way. The teacher should be satisfied if students grasp the main points. The class must not get bogged down on details. This will tend to destroy the desired effect--a comprehension of the extension of the number system to include rational numbers and the need for defining the operations as we do if the commutative, associative, and distributive properties and the properties of zero and one are to be retained.

6-1. An Invitation to Pretend.

Pupils in the seventh grade have had considerable experience in the manipulation of fractions. It is necessary to help them understand that there is more to be learned about rational numbers. The approach here is based upon a frank recognition of the background the pupils have. They are asked to pretend that they are studying rational numbers for the first time and to prepare for a new approach to the topic.

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Answers to Exercises 6-1

1.

÷	1	2	3	4	5	6	7	8
1	1	no	no	no	no	no	no	no
2	2	1	no	no	no	no	no	no
3	3	no	1	no	no	no	no	no
4	4	2	no	1	no	no	no	no
5	5	no	no	no	1	no	no	no
6	6	3	2	no	no	1	no	no
7	7	no	no	no	no	no	1	no
8	8	4	no	2	no	no	no	1

2. 18 bars, 6 bars, . He could not have shared the bars equally unless he broke two of them into smaller pieces. 18 ÷ 4 has no solution in the set of whole numbers.

3. a. This problem is solved by dividing 16 by 2. There are 8 boys on each team.

   b. Again you may start by dividing 17 by 2. You would not say that the answer is $\frac{17}{2}$ or $8\frac{1}{2}$ boys on each team. (This answer would be rather unsatisfactory especially for one of the boys.)

   The problem has no answer since the counting numbers are not closed under division. You do not need to extend the number system to answer questions like this because boys have the property of always occurring in whole number quantities.

   Contrast this problem with Problem 2 where the chocolate bars could be divided.

6-2. The Invention of the Rational Numbers.

The need for numbers other than the whole numbers is emphasized in this section. This need arises when we wish our numbers to serve not only for the purpose of counting but also for the purpose of measuring. This need is established in the example in which a mother wants to divide 100 inches of ribbon equally among her three daughters. She can fold the ribbon and cut it into three pieces of equal length, but if she is restricted to the whole numbers, then she has no number with which to describe this length although its physical existence is apparent. This dilemma is met by inventing a new number which is the result of dividing 100 by 3. Thus, the extension of the number system is motivated.

The teacher should realize that the numbers referred to in this book as the rational numbers are really only the non-negative rational numbers. Since negative numbers are not mentioned in this book it was not considered wise to call our numbers the non-negative rational numbers.

The student may be interested to know that he has seen once before an extension of the number system, i.e., when the set of whole numbers was obtained by the adjoining of the number 0 to the set of counting numbers. He may further be interested in knowing that after three more extensions (which he will see in later years) the number system will be complete for all mathematical purposes. The hierarchy of extensions of the number system is given below:

Counting numbers
Whole numbers
Non-negative rational numbers
Rational numbers
Real numbers
Complex numbers

Three more times in his mathematical experience the student will witness the invention of new numbers, namely:

(1) When the negative rational numbers are adjoined to the non-negative rational numbers to obtain the rational numbers so as to make the number system closed under the operation of subtraction.

(2) When the irrational numbers are adjoined to the rational numbers to obtain the set of real numbers so as to make the number system closed under the operation of taking limits.

(3) When the imaginary numbers are adjoined to the set of real numbers to obtain the set of complex numbers so as to make the number system closed under the operation of taking square roots.

In this section the non-negative rational numbers were adjoined to the whole numbers so as to make the number system closed under division (except by 0). Viewed not just as part of the seventh grade course but as part of one's complete mathematical experience, the presentation in this chapter should help the student to appreciate and understand the logical and systematic development of the hierarchy of number systems.

Answers to Exercises 6-2

1. $\frac{15}{2}$ means 15 divided by 2 or $15 \div 2$.
   $\frac{43}{3}$ means 43 divided by 3 or $43 \div 3$.
   $\frac{17}{17}$ means 17 divided by 17 or $17 \div 17$.
   $\frac{29}{14}$ means 29 divided by 14 or $29 \div 14$.
2. Both blanks can be answered: whole number or counting number.
3. $13 \div 9$, $\frac{13}{9}$, $9\overline{)13}$
4. 250 miles
5. $233\frac{1}{3}$ miles
6. 500 miles
7. a. 64
   b. 55
   c. 54
   d. 89
   e. 91
   f. 77
   g. 91
   h. 111
   i. 77
8. a. About 156
   b. 225

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6-3. Fractions and Rational Numbers.

Fractions are names for rational numbers. Fractions are not the numbers themselves. Fractions are the symbols that we see written on the paper. Rational numbers are abstract mathematical entities that we cannot see. In a similar way love and hate are abstract concepts. It is important in this book to make a careful distinction between numbers and their names. It is especially important to distinguish between rational numbers and fractions.

Failure to distinguish between things and their names sometimes causes confusion. Consider the two sentences:

> Kennedy is the president of the United States.
> Kennedy addressed the convention.

If in the second sentence we replace the word "Kennedy" by the words "the President of the United States" we obtain:

> The President of the United States addressed the convention.

This sentence conveys the same meaning as the second sentence above. No confusion arises.

Now consider the sentences:

> John is a four-letter word.
> John walked down the street.

If, in the second sentence, we replace the word "John" by the words, " a four-letter word" we obtain:

> A four-letter word walked down the street.

This, obviously, does not convey the same meaning as the second sentence above. What causes this confusion? What happens here that didn't happen in the first pair of sentences? The answer is that in the sentence,

> John is a four-letter word,

we were talking about the name of a man, while in the sentence,

> John walked down the street,

we were talking about the man himself. In one case we were talking about a living, breathing, walking human being; in the other case we were talking about marks on a sheet of paper.

Mathematicians have a way of avoiding such confusion. When they are talking about the name itself they put it inside quotation marks. They write the first sentence in the above pair as:

"John" is a four-letter word.

In this sentence we are talking about the actual pencil scratches that appear between the quotation marks. To a certain extent this policy is adopted in this book. We say that the numbers $\frac{3}{2}$ and $\frac{6}{4}$ are the same, but the fractions "$\frac{3}{2}$" and "$\frac{6}{4}$" are not the same. To avoid excessive use of quotation marks, we restrict our use of them to cases in which confusion is likely to arise without them. Moreover, we seldom use quotation marks when the fractions appear on display lines or when we specifically say that we are talking about the fractions (or names) and not about the numbers themselves. Students should not be required to learn how to use quotation marks in this way, but if they inquire about their use in the text, then the teacher may give as much explanation as seems appropriate.

The word "express" is frequently used with the meaning "give a name for." For example, if we were to say "divide 6 by 2", we would expect the answer 3, but if we were to say "divide 6 by 2 and express the answer as a fraction," then we would expect the answer $\frac{6}{2}$ or $\frac{3}{1}$ or even $\frac{12}{4}$, but not 3.

To indicate the way in which fractions are used, the following discussion will be helpful. There are some who may adopt the point of view that $5 + 2$ is an indicated sum which is not yet performed and that

$$5 + 2 = 7$$

means that 7 is the answer to the problem. In this book the attitude is taken that $5 + 2$ is a number and that

$$5 + 2 = 7$$

means that 5 + 2 and 7 are the same number, or in other words "5 + 2" and "7" are different numerals for the same number. If the question is asked, "what is the sum of 5 and 2 ?", then

$$5 + 2$$

is a correct answer to the question. What is really desired when such a question is asked is that the answer should be expressed in a certain form. The ordinary way of expressing numbers in the decimal notation constitutes what mathematicians call a "canonical" form. This means that every member of a certain set (in this case, the whole numbers) can be expressed in this form and in only one way. Thus,

$$5 + 2,\ \frac{14}{2},\ 19 - 12,\ 7,\ 2^2 + 3,\ 12_{five}$$

are all names for the same whole number, but of these names, "7" is the one and only way of expressing this number in the ordinary decimal notation. Therefore, "7" is the canonical form for this number. Similarly, the fractions in simplest form are canonical forms for the rational numbers. That is

$$\frac{6}{4},\ 2 \cdot \frac{3}{4},\ \frac{2}{5} \cdot \frac{15}{4},\ \frac{3}{2},\ \frac{2}{3} + \frac{5}{6}$$

are all names for the same number, but "$\frac{3}{2}$" is the one and only way of expressing this number as a fraction in simplest form. Therefore, "$\frac{3}{2}$" is the canonical form for this number. This terminology is never introduced in the text but it will be helpful to explain it here because this is the kind of thing we are doing.

Similarly, the equation

$$\frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$$

can be interpreted:

$$"\frac{2}{3} \cdot \frac{7}{5}" \text{ and } "\frac{14}{15}"$$

are different names for the same number. Much of the work in the text can then be considered as solving the problem of finding names of a certain type for numbers. In fact, problems are sometimes worded in the form: "multiply $\frac{2}{3}$ by $\frac{7}{5}$ and express the answer as a fraction." To such a problem $\frac{14}{15}$ is then a correct answer but $\frac{2}{3} \cdot \frac{7}{5}$ is not, because although

"$\frac{2}{3}$" and "$\frac{7}{5}$" are both fractions, "$\frac{2}{3} \cdot \frac{7}{5}$" is not a fraction. A fraction, according to the definition in the text, is an expression of the form

$$\frac{a}{b}$$

where a and b are whole numbers with b not equal to zero. A subtle point is that "$\frac{2 \cdot 7}{3 \cdot 5}$" is a fraction and would get by as an answer to the question posed above.

It is probably not necessary to put more stress on these points than is done in the text. It is written here merely to furnish additional background for the teacher.

Perhaps it should be mentioned here that another canonical form for the fraction $\frac{7}{5}$ is $1\frac{2}{5}$. The form $1\frac{2}{5}$ is popularly called a "mixed number". This is unfortunate since if anything is mixed, it is the numeral. The form $1\frac{2}{5}$ is really an indicated sum. It is a short cut for $1 + \frac{2}{5}$. At this time we shall prefer the form $\frac{7}{5}$.

It is suggested that Class Discussion Exercises 6-3b be used in class in preparation for Section 6-4.

Answers to Exercises 6-3a

1. a. Yes
   b. 0
   c. Yes

2. a. No
   b. Yes, a fraction name for this number is $\frac{3}{4}$.

3. No.

4. All of them.

5. a., c., f., are expressed as fractions already. (Fraction names for the others are: b. $\frac{63}{14}$, d. $\frac{8}{7}$, e. $\frac{215}{100}$.)

6. a. $\frac{56}{7}$ b. All of the numerals.

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7. a. $\frac{750}{3}$ mi., 250 mi.

   b. $\frac{90}{3}$ in., 30 in.

   c. $\frac{60}{12}$ cents, 5 cents.

8. a. $\frac{700}{3}$ mi.

   b. $\frac{100}{3}$ in.

   c. $\frac{43}{12}$ cents.

   The questions cannot be answered without fractions and those fractions above are in simplest form, but many pupils will give $233\frac{1}{3}$, $33\frac{1}{3}$, $3\frac{7}{12}$ as answers which, while correct, are not in the form of a fraction. A mixed number like $233\frac{1}{3}$ is actually an indicated sum of a counting number and a fraction: $233 + \frac{1}{3}$. This distinction should be made for the student. (Such forms are explained in Chapter 8.)

9. a. 37, 1, $\frac{111}{3}$, $\frac{7}{7}$

   b. 37, 1, 0, $\frac{0}{3}$, $\frac{111}{3}$, $\frac{7}{7}$

   c. 37, 1, 0, $\frac{2}{3}$, $2\frac{1}{2}$, $\frac{0}{3}$, $\frac{111}{3}$, $\frac{7}{7}$, $\frac{25}{10}$, $\frac{5}{2}$

   d. $\frac{2}{3}$, $\frac{0}{3}$, $\frac{111}{3}$, $\frac{7}{7}$, $\frac{25}{10}$, $\frac{5}{2}$

   e. $\frac{10}{0}$, $\frac{0}{0}$

   f. 37 and $\frac{111}{3}$; 0 and $\frac{0}{3}$; $2\frac{1}{2}$, $\frac{25}{10}$, and $\frac{5}{2}$.

Answers to Exercises 6-3b

1. a. 30 c. 250 e. 480
   b. 5 d. 3 f. division
2. a. 30 c. 250 e. 480
   b. 5 d. 3 f. 91
3. a. 5 c. 8 e. 91 g. 1
   b. 5 d. 3 f. 77 h. 0
4. division

6-4. The Meaning of Division.

This short section is crucial to Section 6-5 and to all the rest of the chapter. Though it is not put in these words, the student is in effect asked to agree that

$$20 \div 4 \quad \text{or} \quad \frac{20}{4}$$

is the solution of the equation

$$4 \cdot x = 20.$$

What is said is that $20 \div 4$ or, what is the same thing, $\frac{20}{4}$ is the number x for which

$$4 \cdot x = 20.$$

The student has already used this fact often in checking division, but he may never have thought of it in this way. Most of the problems in the preceding section have been designed to anticipate this statement.

In the next section this same statement will be made in still another way. In that section it is stated that $\frac{4}{3}$ has the property that

$$3 \cdot \frac{4}{3} = 4 .$$

In other words, the same type of statement as the above is made without using a letter to represent a number. The student must understand this statement when he sees it expressed in either of these two ways.

The problems underscore the one idea of this section.

Note that the idea that $4 \cdot x = 20$ is equivalent to $x = \frac{20}{4}$ is used in later work in ratio and proportion. It enables pupils to solve percent problems.

Answers to Exercises 6-4

1. a. $9 \cdot \frac{8}{9} = 8$
   b. $4 \cdot \frac{7}{4} = 7$
   c. $3 \cdot x = 6$
   d. $2 \cdot x = 5$
   e. $3 \cdot x = 9$
   f. $6 \cdot x = 18$
   g. $4 \cdot x = 10$
   h. $1 \cdot x = 7$

2. c. 2 e. 3 f. 3 h. 7

3. a. $\frac{3}{2}$
   b. $\frac{40}{5}$ or 8
   c. $\frac{15}{4}$
   d. $\frac{56}{7}$ or 8
   e. 7 or $\frac{21}{3}$
   f. $\frac{11}{5}$
   g. 6 or $\frac{60}{10}$
   h. $\frac{1}{8}$

4. a. $3x = 12$; x is the number of cookies each boy receives.
   b. $10x = 160$; x is the number of miles per gallon.
   c. $30x = 20$; x is the number of bags of cement per foot.

5. a. True since $13 \times 13 = 169$
   b. False since $16 \times 17 = 272$
   c. True since $124 \times 6 = 744$
   d. True since $13 \times 11 = 143$
   e. True since $151 \times 101 = 15251$

6-5. Rational Numbers in General.

The student is finally told what

$$\frac{a}{b}$$

means in general. Taken cold, this definition might be quite formidable, but it is hoped that the student will take it in his stride following the great build-up that it has been given.

The horrible example is put in for the purpose of being incomprehensible. It is hoped that this will provide motivation for learning the actual definition which follows. It is hoped if he is faced with the alternatives of understanding this gobbledegook or of understanding what is meant when letters are used to represent numbers then the student will choose the latter. The teacher might point out that before men developed suitable mathematical notation they had to express themselves very much in the style of the horrible example. This had the effect of slowing down progress in mathematics.

The exact wording of the definition requires some analysis. The statement is

Definition: If a and b are whole numbers with b not equal to 0, then

$$\frac{a}{b}$$

is the number x for which $b \cdot x = a$.

What this definition does in effect is to postulate the existence of a solution to the equation.

$$b \cdot x = a$$

for every choice of whole numbers a and b with b not equal to 0. It is evident that this requirement extends our number system beyond what it was before because, for example, the equation

$$4 \cdot x = 3$$

does not have a solution which is a whole number. Since we say that this equation does have a solution, this solution must be something other than a whole number.

A very important word in the definition is the word "the". By saying, "the number x for which $b \cdot x = a$", we require that there be only one number with this property. If we wished to permit the possibility that there could be more than one such number we would have written, "$\frac{a}{b}$ is a number x for which $b \cdot x = a$". This point is not taken up in the text although the fact that there is only one such number is strongly used in the later development. It is felt that the student will most likely not think of the possibility that there is more than one possible value of x. If the possibility were suggested in the text, it might cause serious confusion. The correctness of the development in the later sections consequently rests on the use of the word "the" in this definition. If the question does come up, the teacher should answer it, of course. Perhaps the teacher can illustrate by observing that when we say "a Justice of the Supreme Court" we suggest that there may be more than one Justice, but when we say "the Chief Justice of the Supreme Court" we mean that there is only one Chief Justice.

The last paragraph of this section is devoted to putting the defining property of $\frac{a}{b}$ in the form:

$$\frac{a}{b} \text{ has the property that } b \cdot \frac{a}{b} = a .$$

The exercises provide drill in the use of the definition.

Answers to Exercises 6-5

1. a. $3 \cdot \frac{6}{3} = 6$
   b. $5 \cdot \frac{50}{5} = 50$
   c. $7 \cdot \frac{63}{7} = 63$
   d. $12 \cdot \frac{132}{12} = 132$
   e. $19 \cdot \frac{1729}{19} = 1729$
   f. $35 \cdot \frac{1960}{35} = 1960$

2. a. $\frac{6}{3}$
   b. $\frac{50}{5}$
   c. $\frac{63}{7}$
   d. $\frac{132}{12}$
   e. $\frac{1729}{19}$
   f. $\frac{1960}{35}$

3. a. $x = \frac{7}{3}$
   b. $x = \frac{4}{5}$
   c. $x = \frac{5}{7}$
   d. $x = \frac{6}{12}$
   e. $x = \frac{29}{19}$
   f. $x = \frac{345}{35}$

4. a. $5 \cdot \frac{6}{5} = \underline{6}$
   b. $12 \cdot \frac{7}{12} = \underline{7}$
   c. $9 \cdot \frac{6}{9} = \underline{6}$
   d. $3 \cdot \frac{5}{3} = \underline{5}$
   e. $5 \cdot \frac{10}{5} = \underline{10}$
   f. $14 \cdot \frac{13}{14} = \underline{13}$

5. a. $\underline{5} \cdot \frac{7}{5} = 7$
   b. $\underline{8} \cdot \frac{5}{8} = 5$
   c. $\underline{9} \cdot \frac{3}{9} = 3$
   d. $\underline{5} \cdot \frac{14}{5} = 14$
   e. $\underline{9} \cdot \frac{11}{9} = 11$
   f. $\underline{11} \cdot \frac{9}{11} = 9$

6. In Problems f - j, many answers are possible, but encourage the pupils to give the one which follows from the definition of rational number.

   a. 5

   b. 6

   c. 9

   d. 4

   e. 14

   f. $\frac{9}{11}$

   g. $\frac{1}{16}$

   h. $\frac{1}{4}$

   i. $\frac{2403}{1739}$

   j. $\frac{17963}{609253}$

7. Here, many answers are possible, but insist on the following:

   a. $\underline{5} \cdot \frac{6}{5} = \underline{6}$

   b. $\underline{10} \cdot \frac{9}{10} = \underline{9}$

   c. $\underline{6} \cdot \frac{3}{6} = \underline{3}$

   d. $\underline{16} \cdot \frac{24}{16} = \underline{24}$

   e. $\underline{12} \cdot \frac{1}{12} = \underline{1}$

   f. $\underline{7} \cdot \frac{0}{7} = \underline{0}$

   g. $\underline{1} \cdot \frac{5}{1} = \underline{5}$

   h. $\underline{62973} \cdot \frac{93147}{62973} = \underline{93147}$

8. This can be solved in a fairly systematic way. E times EXAM must equal MATH. E can't be 4 or larger because E times EXAM would be at least a 5-digit number. Since $E > 2$, then $E = 3$. So $M = 9$ and $H = 7$ $(3 \times 9 = 27)$. A can't be 1, 2, 4, 5, 6 by trial and error. But $A = 8$ does work. $9867 = 3 \times (3289)$

6-6. Properties of Operations with Rational Numbers.

Now that the number system has been extended from the counting numbers to the rational numbers and the system is pretty well established, the time is ripe for extending the arithmetic operations so as to operate on the rational numbers. The student must be told that when these operations are so extended, this extension must be made in such a way as to preserve the commutative, associative and distributive properties, the identity properties of 0 and 1 and the multiplication property of 0; $(0 \cdot a = a \cdot 0 = 0)$. Many expert teachers feel that the students will not accept with any conviction the bald statement that the preservation of these properties is desirable. The need for the preservation of these properties is consequently motivated by an analogy. It is hoped that the student will agree that if a concept is extended to have a wider meaning than it originally had, then the extension should be made in such a way as to preserve the properties pertaining to this concept.

The remaining sections of this chapter are devoted to showing how the rules of operation for fractions are consequences of these properties and the definition of the rational numbers, i.e., the relation

$$b \cdot \frac{a}{b} = a.$$

The exercises in this section provide further drill in the use of the definition in 6-5.

Answers to Exercises 6-6

1. There are many possible answers.
   - a. $\left(\frac{2}{3} + \frac{3}{4}\right) + \frac{7}{5} = \frac{2}{3} + \left(\frac{3}{4} + \frac{7}{5}\right)$
   - b. $\frac{2}{3}\left(\frac{3}{4} + \frac{7}{5}\right) = \frac{2}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{7}{5}$
   - c. $\frac{2}{3} \cdot 1 = \frac{2}{3}$
   - d. $\frac{2}{3} \cdot \frac{3}{4} = \frac{3}{4} \cdot \frac{2}{3}$

2. (a) and (b) have many possible answers.
   (c) 0; (d) $\frac{2}{3}$, $\frac{7}{4}$, etc. (e) 1.5, $1\frac{1}{2}$, $2\overline{)3}$, etc.

3. All are rational numbers.

4. a. $x = \frac{14}{15}$
   b. $15 \cdot \frac{14}{15} = 14$
   c. $\underline{15} \cdot \frac{14}{15} = 14$
   d. $15 \cdot \frac{14}{15} = \underline{14}$
   e. $\underline{15} \cdot \frac{14}{15} = \underline{14}$

5. a. $x = \frac{4}{3}$
   b. $3 \cdot \frac{5}{3} = 6$
   c. $4 \cdot \frac{5}{4} = 5$
   d. $7 \cdot \frac{6}{7} = \underline{6}$
   e. $\underline{8} \cdot \frac{5}{8} = 5$
   f. $x = \frac{13}{15}$
   g. $\underline{4} \cdot \frac{7}{4} = \underline{7}$
   h. $x = \frac{5}{6}$
   i. $9 \cdot \frac{7}{9} = 7$
   j. $\underline{11} \cdot \frac{18}{11} = \underline{18}$
   k. $\underline{6} \cdot \frac{1}{6} = \underline{1}$
   l. $5 \cdot \frac{1}{5} = 1$
   m. $x = \frac{1}{5}$
   n. $17 \cdot \frac{0}{17} = 0$

6. a. 2
   b. 4
   c. 5
   d. 5
   e. 7
   f. 7
   g. 6
   h. 1
   i. 8
   j. 3

6-7. Multiplication of Rational Numbers.

The student is reminded that multiplication is to be extended to rational numbers and that it must be defined in a way that preserves the commutative and associative properties of multiplication and the properties of 1 and 0. The only additional information available is that $b \cdot \frac{a}{b} = a$ and that a and b are whole numbers $(b \neq 0)$ so that multiplication of a and b is familiar.

Multiplication of the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is defined as $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ in a series of steps in which the pupil is asked to answer questions and then compare his responses with the correct ones supplied on the same page. He is asked to cover the printed response until after he has produced his own.

The steps are written in a modified "program" form to guide the pupil. It is recommended that each question be discussed in class as it appears to need clarification.

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1. We do not know that the product of $\frac{2}{3}$ and $\frac{7}{5}$ exists or that $\frac{2}{3} \cdot \frac{7}{5}$ has meaning. It is our hope that $\frac{2}{3} \cdot \frac{7}{5}$ does have meaning and that we can find a way to express it as a fraction.

2.
3. Here we assume that $\frac{2}{3} \cdot \frac{7}{5}$ is a number and forge ahead. We know that $3 \cdot \frac{2}{3} = 2$ and that $5 \cdot \frac{7}{5} = 7$; and we also know the properties of multiplication which we wish to retain as we extend multiplication to rational numbers.

4.
5. This seems to be a difficult idea although we have worked zealously to prepare for it. The fact is that a number has many possible numerals. Two specific numerals for one number in our discussion are "$3 \cdot \frac{2}{3}$" and "2". Also two numerals for the other number are "$5 \cdot \frac{7}{5}$" and "7".

6. We form the product of the two numbers. This product can be named in two ways employing the pairs of names we have been talking about.

   One way to name the product: $(3 \cdot \frac{2}{3}) \cdot (5 \cdot \frac{7}{5})$
   Another way to name the product: $2 \cdot 7$
   In both cases the product is the same number:

   $$(3 \cdot \tfrac{2}{3}) \cdot (5 \cdot \tfrac{7}{5}) = 2 \cdot 7$$

7. It seems reasonable at this stage to use "14" for "$2 \cdot 7$" although "$2 \cdot 7$" can be carried through the argument just as well.

8. We require the associative and commutative properties of multiplication to continue to be properties of multiplication as we extend the operation to rational numbers. The factors on the left are reordered and regrouped by means of the commutative and associative properties of multiplication. This is the key step in the demonstration. It is here that the properties of multiplication are used in establishing the result, thus achieving the purpose of the section and, in fact,

the purpose of the chapter. The student realizes that factors in a product can be reordered and regrouped by use of the associative and commutative properties (this was gone into in some detail in Chapter 3). He will probably accept this step without reservation. Actually this step masks some difficulties. Here is the shortest way of showing that

$$(3 \cdot \tfrac{2}{3}) \cdot (5 \cdot \tfrac{7}{5}) = (3 \cdot 5) \cdot (\tfrac{2}{3} \cdot \tfrac{7}{5})$$

using one property at a time.

$$\begin{aligned}(3 \cdot \tfrac{2}{3}) \cdot (5 \cdot \tfrac{7}{5}) &= [(3 \cdot \tfrac{2}{3}) \cdot 5] \cdot \tfrac{7}{5} && \text{associative}\\ &= [5 \cdot (3 \cdot \tfrac{2}{3})] \cdot \tfrac{7}{5} && \text{commutative}\\ &= [(5 \cdot 3) \cdot \tfrac{2}{3}] \cdot \tfrac{7}{5} && \text{associative}\\ &= [(3 \cdot 5) \cdot \tfrac{2}{3}] \cdot \tfrac{7}{5} && \text{commutative}\\ &= (3 \cdot 5) \cdot (\tfrac{2}{3} \cdot \tfrac{7}{5}) && \text{associative}\end{aligned}$$

Most students would be confused if these details were brought in.

9. We use "15" for "$3 \cdot 5$". See Step 7 above.
10. We examine $15 \cdot (\frac{2}{3} \cdot \frac{7}{5}) = 14$. If this were $15 \cdot x = 14$ we should recognize immediately that $x = \frac{14}{15}$. But x is $(\frac{2}{3} \cdot \frac{7}{5})$.

    Therefore $\frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$ and we have found a fraction to name the product.

A pupil may point out that the numbers $\frac{2}{3}$ and $\frac{7}{5}$ have many names and he may ask how we can be sure that the result of using different names will be a name for the number $\frac{14}{15}$. You might use another fraction for $\frac{2}{3}$, such as $\frac{22}{33}$ and show that $\frac{22}{33} \cdot \frac{7}{5} = \frac{154}{165} = \frac{14 \cdot 11}{15 \cdot 11} = \frac{14}{15}$. The trouble is that simplification of fractions, or "equivalent fractions", is dealt with in Section 6-8.

The point to be made is that this is the only possible definition for the product of two rational numbers, which is consistent with the properties of multiplication.

The significance of the word "the" in the definition of rational numbers in Section 6-5 becomes apparent. Steps 10 and 11 use the definition that in $b \cdot x = a$, x is the number $\frac{a}{b}$. It is shown that n is the number $\frac{2 \cdot 7}{3 \cdot 5}$ and also is $(\frac{2}{3} \cdot \frac{7}{5})$. Therefore, $\frac{2 \cdot 7}{3 \cdot 5}$ and $\frac{2}{3} \cdot \frac{7}{5}$ must be the same number.

Not only does this result show us the procedure to follow with the symbols (fractions) when we multiply the numbers, but it tells us that the product of two rational numbers is also a rational number. The product has a fraction name which establishes that the product is a rational number. The set of rational numbers is closed under multiplication.

Nothing is gained by restricting the words numerator and denominator to mean the numerals which form a fraction. Hence the students are told to use these words to mean numbers or numerals, whichever is convenient.

Answers to Exercises 6-7b

1. a. $\frac{10}{63}$
   b. $\frac{2}{9}$
   c. $\frac{0}{40}$
   d. $\frac{30}{20}$
   e. $\frac{1}{30}$
   f. $\frac{27}{32}$
   g. $\frac{10}{15}$
   h. $\frac{6}{5}$
   i. $\frac{0}{21}$
   j. $\frac{1288}{1140}$
   k. $\frac{1288}{1140}$
   l. $\frac{x \cdot z}{y \cdot w}$

2. a. $\frac{12}{12}$
   b. $\frac{30}{30}$
   c. $\frac{5}{5}$
   d. $\frac{28}{28}$
   e. $\frac{6}{6}$
   f. $\frac{a \cdot b}{b \cdot a}$ or $\frac{a \cdot b}{a \cdot b}$

3. a. $\frac{15}{20}$
   b. $\frac{6}{21}$
   c. $\frac{27}{6}$
   d. $\frac{8}{7}$
   e. $\frac{c}{d}$

4. a. $\frac{5}{7}$
   b. $\frac{2}{3}$
   c. $\frac{3}{4}$
   d. $\frac{1}{1}$
   e. $\frac{5}{6}$
   f. $\frac{2}{2}$

5. $\frac{15}{77}$

6. $\frac{3}{40}$

7. $\frac{32}{105}$

8. $\frac{25}{56}$

Answers to Exercises 6-7c

1. Commutative property of multiplication.

2. First write the whole number as a fraction with denominator 1, and then write the product as a fraction where its numerator is the product of the numerators and its denominator is the product of the denominators.

3. a. $\frac{15}{4}$
   b. $\frac{6}{7}$
   c. $\frac{8}{15}$
   d. $\frac{0}{8}$
   e. $\frac{12}{5}$
   f. $\frac{88}{9}$
   g. $\frac{8}{5}$
   h. $\frac{8}{3}$
   i. $\frac{0}{13}$

4. a. $7 \cdot \frac{1}{6}$
   b. $2 \cdot \frac{1}{5}$
   c. $3 \cdot \frac{1}{4}$
   d. $4 \cdot \frac{1}{9}$
   e. $3 \cdot \frac{1}{3}$
   f. $6 \cdot \frac{1}{9}$

5. $\frac{36}{7}$

6-8. Equivalent Fractions.

The word "simplify" has been used in conformity with the definition of "simplest form". The word "cancel" has been avoided because it is associated with a mechanical deletion of similar factors.

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The rule

$$\frac{a}{b} = \frac{a \cdot k}{b \cdot k}$$

is established by introducing a concrete example. The result is shown to follow from the identity property of 1 and the product of rational numbers. The student is given a logical basis for simplifying fractions and writing equivalent fractions.

Common denominators and least common denominators are also discussed in this section to provide drill in writing equivalent fractions and to prepare for the extensive use of these notions in the addition and subtraction of fractions and in Chapter 8.

The teacher will note that the mechanics involve a three-step procedure with each step supported by a logical reason.

Answers to Exercises 6-8a

1. 6

2. $a = 4$, $b = 3$, $k = 6$

3. Because the numerator and denominator have no common factor larger than 1.

4. a. $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}$, etc.
   b. $\frac{4}{10}, \frac{6}{15}, \frac{10}{25}$, etc.
   c. $\frac{1}{1}, \frac{2}{2}, \frac{101}{101}$, etc.
   d. $\frac{3}{1}, \frac{6}{2}, \frac{9}{3}$, etc.
   e. $\frac{0}{1}, \frac{0}{24}, \frac{0}{101}$, etc.
   f. $\frac{5}{6}, \frac{10}{12}, \frac{100}{120}$, etc.
   g. $\frac{2}{1}, \frac{24}{12}, \frac{4}{2}$, etc.
   h. $\frac{6}{5}, \frac{12}{10}, \frac{18}{15}$, etc.
   i. $\frac{2}{2}, \frac{3}{3}, \frac{4}{4}$, etc.
   j. $\frac{9}{27}, \frac{3}{9}, \frac{1}{3}$, etc.

5. a. $2 \cdot 3 \cdot 5$ or 30
   b. $\frac{3}{10} = \frac{9}{30}, \frac{4}{15} = \frac{8}{30}$

6. a. $\frac{2}{3} = \frac{10}{15}$
   b. $\frac{7}{5} = \frac{21}{15}$
   c. $\frac{11}{37} = \frac{33}{111}$
   d. $\frac{2}{3} = \frac{74}{111}$
   e. $\frac{1}{2} = \frac{9}{18}$
   f. $\frac{1}{1} = \frac{18}{18}$

7. a. $\frac{10}{15}, \frac{21}{15}$
   b. $\frac{9}{30}, \frac{8}{30}$
   c. $\frac{12}{42}, \frac{35}{42}$
   d. $\frac{33}{111}, \frac{74}{111}$
   e. $\frac{5}{20}, \frac{4}{20}$

Answers to Exercises 6-8b

1. a. $\frac{9}{21} = \frac{3 \cdot 3}{7 \cdot 3} = \frac{3}{7}$
   b. $\frac{14}{16} = \frac{7 \cdot 2}{8 \cdot 2} = \frac{7}{8}$
   c. $\frac{12}{24} = \frac{1 \cdot 12}{2 \cdot 12} = \frac{1}{2}$
   d. $\frac{15}{45} = \frac{1 \cdot 15}{3 \cdot 15} = \frac{1}{3}$
   e. $\frac{24}{42} = \frac{4 \cdot 6}{7 \cdot 6} = \frac{4}{7}$
   f. $\frac{55}{121} = \frac{5 \cdot 11}{11 \cdot 11} = \frac{5}{11}$
   g. $\frac{40}{60} = \frac{2 \cdot 20}{3 \cdot 20} = \frac{2}{3}$
   h. $\frac{75}{100} = \frac{3 \cdot 25}{4 \cdot 25} = \frac{3}{4}$
   i. $\frac{123}{321} = \frac{41 \cdot 3}{107 \cdot 3} = \frac{41}{107}$
   j. $\frac{111}{37} = \frac{3 \cdot 37}{1 \cdot 37} = 3$
   k. $\frac{432}{234} = \frac{24 \cdot 18}{13 \cdot 18} = \frac{24}{13}$
   l. $\frac{111111}{1111} = \frac{10101 \cdot 11}{101 \cdot 11} = \frac{10101}{101}$.

2. a. $\frac{6}{3}$
   b. $\frac{15}{3}$
   c. $\frac{6}{15}$
   d. $\frac{14}{35}$
   e. $\frac{12}{30}$
   f. $\frac{5}{15}$

3. a. $\frac{2}{4}, \frac{3}{4}$
   b. $\frac{14}{28}, \frac{7}{28}$
   c. $\frac{14}{60}, \frac{9}{60}$
   d. $\frac{16}{20}, \frac{25}{20}$
   e. $\frac{16}{4}, \frac{1}{4}$
   f. $\frac{15}{5}, \frac{1}{5}$
   g. $\frac{15}{5}, \frac{1}{5}$
   h. $\frac{4}{484}, \frac{11}{484}$
   i. $\frac{15}{30}, \frac{20}{30}, \frac{18}{30}$
   j. $\frac{5}{30}, \frac{4}{30}, \frac{9}{30}$

4. a. 2
   b. 4
   c. $\frac{3}{2}$
   d. $\frac{4}{5}$
   e. $\frac{1}{2}$
   f. 1
   g. $\frac{6}{5}$
   h. $\frac{7}{5}$
   i. $\frac{28}{27}$
   j. $\frac{7}{64}$
   k. $\frac{3}{5}$
   l. $\frac{1}{2}$
   m. $\frac{25}{8}$
   n. $\frac{5}{16}$
   o. $\frac{27}{4}$

5. Each son received $\frac{20}{7}$ acres. The total number of acres the sons receive is $4 \cdot \frac{20}{7} = \frac{80}{7}$.

6. 35 cents $\left\{\begin{array}{l}\text{one ounce costs } \frac{80}{16} \text{ cents} \\ \text{7 ounces cost } 7 \cdot \frac{80}{16} \text{ cents or } \frac{560}{16} \text{ cents} \\ \text{or 35 cents .}\end{array}\right.$

7. \$216.25 $\left\{\begin{array}{l}\text{Pay for 1 hour is } \frac{173}{40} \text{ dollars.} \\ \text{Pay for 50 hours is } 50 \cdot \frac{173}{40} \text{ dollars or} \\ \frac{865}{4} \text{ dollars or } 216\frac{1}{4} \text{ dollars.}\end{array}\right.$

8. There are four answers, $\frac{12}{21}$, $\frac{24}{42}$, $\frac{36}{63}$, $\frac{48}{84}$.

6-9. Reciprocals.

Students come to the seventh year with the ability to perform the process of division with rational numbers. They know the rule that division is performed by multiplying the dividend by the reciprocal of the divisor. The introduction of the reciprocal at this point enables the teacher to show that the procedure already known is consistent with the treatment of rational numbers in this chapter.

Answers to Exercises 6-9

1. $\frac{a \cdot b}{b \cdot a} = \frac{a \cdot b}{a \cdot b}$ by commutative property of multiplication.

   $\frac{a \cdot b}{a \cdot b}$ means $a \cdot b \div a \cdot b$ which is 1.

2. Interchange the numerator and denominator of the fraction.

3. a. $\frac{8}{5}$
   b. $\frac{3}{7}$
   c. $\frac{5}{8}$
   d. $\frac{7}{3}$
   e. $\frac{6}{17}$
   f. $\frac{4}{9}$
   g. $\frac{1}{6}$
   h. $\frac{6}{1}$
   i. $\frac{7}{11}$
   j. $\frac{3}{2}$
   k. $\frac{4}{3}$
   l. $\frac{9}{6}$
   m. $\frac{1}{1}$
   n. $\frac{15}{14}$
   o. $\frac{20}{21}$
   p. $\frac{1492}{1729}$

4. Since zero times any rational number is zero, there is no number whose product with zero will be one.

6-10. Division of Rational Numbers.

The pupil is reminded that in the case of division of whole numbers, we had a problem. We could not always divide a whole number by another whole number (always excluding zero as a divisor). To solve this problem we had to turn to the system of rational numbers. Now we are operating with rational numbers. We have defined multiplication of rational numbers ($\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ $b \neq 0,\ d \neq 0$) and the definition of division is our next step.

We may wonder whether rational numbers are closed under division or if an extension of the number system is necessary again. What do we intend to mean by such an expression as

$$\frac{2}{3} \div \frac{5}{7}\ ?$$

We can rephrase the question and ask,

"For what number n is $\frac{2}{3} \div \frac{5}{7} = n$?"

If there is such a number, n, we should like to know what it is. We choose to investigate by requiring the properties listed on page 200 in the Student's Text. We recall that

$a \div b = x$ is equivalent to, or means $a = b \cdot x$

and $\frac{2}{3} \div \frac{5}{7} = n$ is equivalent to, or means $\frac{2}{3} = \frac{5}{7} \cdot n$.

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Thus the question that we seek to answer is restated in terms of multiplication which we have defined for all rational numbers. We now have the question,

"What is the number n for which $\frac{2}{3} = \frac{5}{7} \cdot n$?"

To prepare the pupil for a later step, the text reminds him that the product of $\frac{5}{7}$ and its reciprocal $\frac{7}{5}$ is 1. Then follows the critical step.

In order to help the pupil see how

$$\frac{7}{5} \cdot \frac{2}{3} = \frac{7}{5} \cdot \left(\frac{5}{7} \cdot n\right)$$

follows from

$$\frac{2}{3} = \frac{5}{7} \cdot n$$

it is necessary to make two observations. The first of these observations involves what we mean when we say that two numbers are equal. When we write,

$$6 = \mathrm{VI}$$

for example, we mean that 6 and VI are the same number. That is to say that the symbols "6" and "VI" are different names for the same number.

The second observation concerns the operations of multiplication, addition, etc. These operations are operations on numbers and not on symbols. The expressions,

"$2 \cdot 6$" and "$\mathrm{II} \cdot \mathrm{VI}$"

for example, are two different ways of expressing the product of the same pair of numbers. The products depend only on the numbers which are multiplied, not on the way we choose to name them. Two people, asked to write an expression for the product of two and six might offer the written expressions, "$2 \cdot 6$" and "$\mathrm{II} \cdot \mathrm{VI}$", but both are producing products of the same pair of numbers. The numbers named by "$2 \cdot 6$" and "$\mathrm{II} \cdot \mathrm{VI}$" are the same.

This means that

$$2 \cdot 6 = \mathrm{II} \cdot \mathrm{VI} .$$

Similarly, if $\frac{2}{3} = \frac{5}{7} \cdot n$

then "$\frac{2}{3}$" and "$\frac{5}{7} \cdot n$" are names for the same number and

$$\text{"}\frac{7}{5} \cdot \frac{2}{3}\text{"} \quad \text{and} \quad \text{"}\frac{7}{5} \cdot \left(\frac{5}{7} \cdot n\right)\text{"}$$

are two ways of expressing the product of the same pair of numbers. The first factors in the separate expressions are obviously the same. The second factor is designated in one case by "$\frac{2}{3}$" and in the other by "$\frac{5}{7} \cdot n$". The second factors are the same. Then

$$\frac{7}{5} \cdot \frac{2}{3} = \frac{7}{5} \cdot \left(\frac{5}{7} \cdot n\right)$$

When this principle is understood, there is no need for such rules as "When equals are multiplied by equals, the products are equal" or "When equals are added to equals, the sums are equal."

After we complete this step the rest seems easy. We write $\frac{14}{15}$ for $\frac{7}{5} \cdot \frac{2}{3}$ by means of our definition of multiplication of rational numbers and we require the associative property of multiplication. We then have $\left(\frac{7}{5} \cdot \frac{5}{7}\right) \cdot n$ which becomes $1 \cdot n$ by the property of reciprocals. In the next step we require the identity property of 1 to permit us to move from $1 \cdot n$ to n.

This establishes that $n = \frac{14}{15}$ and finally that

$$\frac{2}{3} \div \frac{5}{7} = \frac{14}{15}$$

which we write as

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5}$$

and thus we can define division by a rational number as multiplication by its reciprocal.

The steps are:

$\frac{2}{3} = \frac{5}{7} \cdot n$	From $\frac{2}{3} \div \frac{5}{7} = n$
$\frac{7}{5} \cdot \frac{2}{3} = \frac{7}{5} \cdot (\frac{5}{7} \cdot n)$	Multiplication of the same number by $\frac{7}{5}$
$\frac{14}{15} = \frac{7}{5} \cdot (\frac{5}{7} \cdot n)$	Product of rational numbers
$\frac{14}{15} = (\frac{7}{5} \cdot \frac{5}{7}) \cdot n$	Associative property
$\frac{14}{15} = 1 \cdot n$	Property of reciprocals
$\frac{14}{15} = n$	Identity property of 1
$\frac{2}{3} \div \frac{5}{7} = n$	First statement
$\frac{2}{3} \div \frac{5}{7} = \frac{14}{15}$	n is $\frac{14}{15}$
$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5}$	Product of rational numbers

This definition of division of rational numbers is the only definition possible consistent with the properties which we require. One more step remains. We have established the following:

If there is a number n for which $\frac{2}{3} = \frac{5}{7} \cdot n$,
then this number n is $\frac{14}{15}$.

We can say that if anything will work, then $\frac{14}{15}$ will. It may be that nothing will work. Perhaps there is no rational number which is a solution to the equation. We must make sure that $\frac{14}{15}$ satisfies the equation:

$$\frac{2}{3} = \frac{5}{7} \cdot n$$

We try $\frac{14}{15}$: $\quad \frac{5}{7} \cdot n = \frac{5}{7} \cdot \frac{14}{15} = \frac{5}{7} \cdot \frac{7 \cdot 2}{5 \cdot 3} = \frac{5}{7} \cdot (\frac{7}{5} \cdot \frac{2}{3})$

$$= (\frac{5}{7} \cdot \frac{7}{5}) \cdot \frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{2}{3}.$$

and find that $\frac{5}{7} \cdot \frac{14}{15}$ is $\frac{2}{3}$. The argument is complete.

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Answers to Exercises 6-10a

	(a)	(b)	(c)
1.	$\frac{14}{15}$	$\frac{14}{15}$	$\frac{6}{35}$
2.	$\frac{15}{14}$	$\frac{15}{14}$	$\frac{35}{6}$
3.	$\frac{21}{10}$	$\frac{35}{6}$	$\frac{21}{10}$
4.	$\frac{10}{21}$	$\frac{6}{35}$	$\frac{10}{21}$
5.	$\frac{10}{21}$	$\frac{6}{35}$	$\frac{10}{21}$
6.	$\frac{21}{10}$	$\frac{35}{6}$	$\frac{21}{10}$
7.	$\frac{15}{14}$	$\frac{15}{14}$	$\frac{35}{6}$
8.	$\frac{14}{15}$	$\frac{14}{15}$	$\frac{6}{35}$
9.	$\frac{2}{5}$	2	10
10.	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{7}{3}$
11.	$\frac{3}{4}$	1	$\frac{65}{66}$
12.	$\frac{8}{7}$	$\frac{3}{4}$	$\frac{20}{21}$
13.	30		
14.	6		
15.	25		

Answers to Exercises 6-10b

1. a. $\frac{1}{11}$ c. $\frac{1}{47}$ e. $\frac{7}{2}$ g. $\frac{7}{1000}$

   b. $\frac{1}{201}$ d. 5 f. $\frac{3}{50}$ h. $\frac{175}{346}$

2. a. $\frac{1}{m}$ d. $\frac{s}{r}$

   b. $\frac{1}{s}$ e. $\frac{w}{t}$

   c. c

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3. Set of reciprocals is $\{1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}.\}$

4. a. $\frac{1}{3}$
   b. 7
   c. $\frac{3}{2}$
   d. $\frac{5}{7}$
   e. $\frac{1}{100}$
   f. $\frac{3}{100}$
   g. $\frac{5}{2}$

5. $\frac{4}{9}$
6. 7
7. 1
8. $\frac{1}{2}$
9. 0
10. 1
11. 15
12. $\frac{2}{7}$
13. $\frac{22}{5}$
14. $\frac{16}{15}$
15. $\frac{44}{21}$
16. $\frac{3}{4}$
17. $\frac{3}{2}$
18. $\frac{10}{3}$
19. $\frac{18}{35}$
20. $\frac{9}{10}$
21. $\frac{24}{25}$

Answers to Exercises 6-10c

1. a. $\frac{1}{1}$
   b. $\frac{72}{1}$
   c. $\frac{25}{81}$
   d. $\frac{1}{2}$
   e. $\frac{1}{12}$
   f. $\frac{3}{2}$
2. a. $\frac{4}{3}$ b. $\frac{3}{4}$ Division is not commutative.
3. a. $\frac{7}{9}$ b. $\frac{4}{7}$ Division is not associative.
4. 14
5. 6
6. $\frac{1}{16}$

6-11. Addition of Rational Numbers.

In this section, pupils learn that the familiar rules for addition of rational numbers are consequences of the distributive property. When the addends are expressed as fractions having the same denominator, the distributive property and the product of rational numbers are the only tools used. When denominators are different it is necessary to invoke equivalent fractions to obtain a common denominator.

The addition of rational numbers expressed as mixed numbers follows the general theory developed in this section. It is treated in detail in Chapter 8.

Answers to Exercises 6-11a

1. The sum can be expressed as a fraction with the same denominator and the numerator will be the sum of the numerators of the two fractions.

2. $\frac{p + q}{s}$

3. a. $\frac{4}{5}$
   b. $\frac{7}{10}$
   c. $\frac{2}{3}$
   d. $\frac{7}{9}$
   e. $\frac{8}{7}$
   f. $\frac{9}{8}$
   g. $\frac{11}{3}$
   h. $\frac{5}{9}$

4. a. $\frac{5}{5}$ or $\frac{1}{1}$
   b. $\frac{8}{10}$ or $\frac{4}{5}$
   c. $\frac{8}{100}$ or $\frac{2}{25}$
   d. $\frac{10}{10}$ or $\frac{1}{1}$
   e. $\frac{6}{9}$ or $\frac{2}{3}$
   f. $\frac{6}{8}$ or $\frac{3}{4}$
   g. $\frac{7}{7}$ or $\frac{1}{1}$
   h. $\frac{21}{7}$ or $\frac{3}{1}$

5. a. 30
   b. 18
   c. 8
   d. 60
   e. 12
   f. 100
   g. 24
   h. 30

6. b. $\frac{4}{18}, \frac{21}{18}$
   c. $\frac{7}{8}, \frac{22}{8}$
   d. $\frac{85}{60}, \frac{24}{60}$
   e. $\frac{28}{12}, \frac{45}{12}$
   f. $\frac{58}{100}, \frac{15}{100}$
   g. $\frac{46}{24}, \frac{27}{24}$
   h. $\frac{35}{30}, \frac{4}{30}$

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Answers to Exercises 6-11b

1. a. $\frac{29}{35}$
   b. $\frac{24}{35}$
   c. $\frac{5}{6}$
   d. $\frac{13}{30}$
   e. $\frac{22}{21}$
   f. $\frac{25}{12}$
   g. $\frac{5}{12}$
   h. $\frac{1}{2}$
   i. $\frac{5}{4}$
   j. $\frac{2}{3}$
   k. $\frac{16}{3}$
   l. $\frac{1}{6}$

2. a. $b \cdot d$
   b. $\frac{a}{b} = \frac{a \cdot d}{b \cdot d}$
   c. $\frac{c}{d} = \frac{b \cdot c}{b \cdot d}$
   d. $\frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$

3. a. $\frac{7}{10}$
   b. $\frac{29}{35}$
   c. $\frac{22}{21}$
   d. $\frac{5}{12}$
   e. $\frac{25}{12}$
   f. $\frac{1}{2}$
   g. $\frac{5}{4}$
   h. $\frac{2}{3}$
   i. $\frac{7}{18}$
   j. $\frac{17}{18}$
   k. $\frac{17}{66}$
   l. $\frac{1}{6}$
   m. $\frac{16}{75}$
   n. $\frac{37}{6}$
   o. $\frac{38}{9}$

4. a. $\frac{7}{3}$
   b. $\frac{7}{3}$
   c. $\frac{15}{8}$
   d. $\frac{15}{8}$

5. $\frac{27}{24}$ lb. or $\frac{9}{8}$ lb.

6. $\frac{13}{24}$ mi.

7. a. $\frac{7}{12}$
   b. $\frac{5}{12}$

8. a. $\frac{5}{6}$
   b. $\frac{1}{6}$

9. $\frac{1}{20}$

10. In this magic square, the sum of each column, each row, and each diagonal is $\frac{45}{8}$.

6-12. Subtraction of Rational Numbers.

Just as division is the inverse operation of multiplication, subtraction is the inverse operation of addition. Subtraction is approached through the fact that

$$a - b$$

is the number x for which

$$b + x = a .$$

A discussion of this fact precedes the derivation of the subtraction procedure.

Answers to Exercises 6-12a

1. a. $\frac{3}{7}$
   b. $\frac{18}{7}$
   c. $\frac{2}{5}$
   d. $\frac{5}{9}$
   e. $\frac{2}{9}$
   f. 1
   g. 0
   h. $\frac{1}{2}$
   i. $\frac{10}{3}$

2. a. $\frac{2}{11}$
   b. $\frac{39}{5}$
   c. $\frac{2}{7}$
   d. $\frac{0}{3}$
   e. $\frac{6}{14}$ or $\frac{3}{7}$
   f. $\frac{14}{14}$ or $\frac{1}{1}$
   g. $\frac{1}{6}$
   h. $\frac{2}{6}$ or $\frac{1}{3}$
   i. $\frac{3}{20}$

3. a. Find a common denominator (or a common multiple of the denominators) for the fractions.
   b. Any common multiple of the denominators.
   The least common multiple of the denominators.
   c. $\frac{7}{8} = \frac{35}{40}$ $\quad$ $\frac{3}{5} = \frac{24}{40}$
   d. $\frac{11}{40}$

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4. a. $\frac{11}{40}$
b. $\frac{1}{6}$
c. $\frac{2}{6}$
d. $\frac{3}{20}$
e. $\frac{2}{9}$
f. $\frac{4}{9}$
g. $\frac{1}{2}$
h. $\frac{8}{3}$
i. $\frac{4}{5}$
j. $\frac{8}{9}$
k. $\frac{1}{15}$
l. $\frac{20}{63}$

5. a. $\frac{1}{6}$
b. $\frac{4}{15}$
c. $\frac{13}{15}$
d. $\frac{13}{35}$
e. $\frac{1}{55}$
f. $\frac{17}{35}$
g. $\frac{3}{10}$
h. $\frac{17}{21}$
i. $\frac{1}{14}$
j. $\frac{1}{6}$
k. $\frac{21}{70}$ or $\frac{3}{10}$
l. $\frac{14}{105}$ or $\frac{2}{15}$

*6. a. $\frac{a}{b} = \frac{a \cdot d}{b \cdot d}$
$\frac{c}{d} = \frac{b \cdot c}{b \cdot d}$

b. $\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d}{b \cdot d} - \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d - b \cdot c}{b \cdot d}$

Answers to Exercises 6-12b

1. $\frac{1}{42}$
2. $\frac{1}{6}$
3. $\frac{37}{6}$
4. $\frac{23}{15}$
5. $\frac{8}{15}$
6. $\frac{1}{6}$
7. $\frac{4}{21}$
8. $\frac{1}{9}$
9. $\frac{7}{132}$
10. $\frac{9}{16}$
11. $\frac{1}{36}$

12. $\frac{7}{12}$ yd. which is more than $\frac{1}{2}$ yd.

6-14. Chapter Review.

Answers to Exercises 6-14

1. a. 3
b. 3
c. 5
d. $4 \cdot \frac{5}{4} = 5$
e. $\frac{5}{6}$
f. $\frac{3}{4}$

2. a. $\frac{1}{3}$
b. $\frac{1}{2}$
c. 2
d. $\frac{4}{5}$
e. $\frac{6}{35}$
f. $\frac{10}{9}$
g. $\frac{8}{7}$
h. 8
i. $\frac{341}{35}$
j. $\frac{8}{5}$
k. $\frac{1}{15}$
l. 1
m. 1
n. $\frac{8}{9}$
o. $\frac{1}{4}$

3. a. $\frac{1}{2}$
b. 1
c. $\frac{16}{49}$
d. $\frac{5}{6}$
e. $\frac{3}{5}$
f. $\frac{3}{4}$
g. $\frac{8}{9}$
h. $\frac{15}{16}$
i. $\frac{4}{9}$
j. $\frac{2}{3}$
k. $\frac{3}{2}$
l. $\frac{3}{8}$
m. $\frac{35}{2}$
n. $\frac{3}{40}$
o. $\frac{5}{36}$
p. 16

4. a. $\frac{5}{6}$
b. $\frac{7}{12}$
c. $\frac{23}{30}$
d. $\frac{17}{15}$
e. $\frac{13}{21}$
f. $\frac{5}{8}$
g. $\frac{5}{4}$
h. $\frac{67}{30}$
i. $\frac{7}{8}$
j. $\frac{17}{6}$
k. $\frac{17}{6}$
l. $\frac{23}{4}$

5. a. $\frac{1}{3}$
b. $\frac{1}{2}$
c. $\frac{5}{42}$
d. $\frac{1}{3}$
e. $\frac{2}{9}$
f. $\frac{13}{42}$
g. $\frac{5}{6}$
h. $\frac{13}{30}$
i. $\frac{71}{12}$
j. $\frac{1}{5}$
k. $\frac{4}{3}$
l. $\frac{40}{99}$

6. a. 1
b. $\frac{2}{3}$
c. $\frac{4}{75}$
d. $\frac{9}{14}$
e. $\frac{5}{4}$
f. $\frac{1}{6}$
g. $\frac{3}{20}$
h. $\frac{3}{5}$
i. $\frac{19}{12}$
j. $\frac{1}{12}$
k. $\frac{1}{14}$
l. $\frac{4}{5}$
m. $\frac{17}{12}$
n. $\frac{2}{3}$
o. $\frac{43}{42}$

7. $\frac{15}{32}$ in.

8. $\frac{2}{9}$

9. $\frac{11}{20}$ mi.

10. $\frac{1}{4}$ cup

11. $\frac{1}{8}$

12. 114

13. $\frac{63}{2}$ qts. or $31\frac{1}{2}$ qts.

14. Multiplication property of one.

15. 30

16. They are called reciprocals of each other.

17. Equivalent fractions will have the same simplest form. (i.e., numerator and denominator have no common factor except 1.) So simplify each fraction.

18. $\underline{\frac{3}{2}}$ is $\underline{2}$.

19. $\underline{2}$ by $\frac{1}{3}$.

20. $\frac{3}{2}$

6-15. Cumulative Review.

Answers to Exercises 6-15

1. a. $\frac{8}{27}$

   b. either $\frac{2^3}{3^3}$ or $(\frac{2}{3})^3$

2. a. $5^3 = 125$ $3^5 = 243$

   b. 3^5 is greater than 5^3 by 118.

3. {0,10,20,30,40,...}. This set is closed under addition.

4. a. 7

   b. No whole number.

   c. 0, 1, 2, 3, 4, 5, or 6

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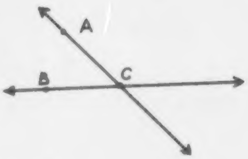
5. True. Call the point of intersection C. Choose a point A on one line and a point B on the other. A, B, and C lie in the same plane since any three points not on the same line are in only one plane. All the points of $\overleftrightarrow{AC}$ and all the points of $\overleftrightarrow{BC}$ lie in this same plane. (If a line contains two different points of a plane, it lies in the plane.)



6. a. point E b. point F c. The empty set.

7. two (i.e., 2, 5)

8. a. It is divisible by 3 since 1 + 0 + 1 + 0 + 1 is divisible by 3.
   b. No, since 1 + 0 + 1 + 0 + 1 is not divisible by 9.

9. 20

10. 2 and 5

Sample Test Questions for Chapter 6

Teachers should construct their own tests, using carefully selected items from those given here and from their own. There are too many questions here for one test. Careful attention should be given to difficulty of items and time required to complete the test.

I. True-False Questions

(T) 1. The product $\frac{3}{7} \cdot \frac{3}{3}$ is equal to $\frac{3}{7}$.

(F) 2. Whole numbers are not rational numbers.

(F) 3. $\frac{5}{7} + \frac{7}{5} = \frac{12}{12}$

(T) 4. In adding rational numbers, if the denominators of the fractions are equal we add the numerators.

(T) 5. The following numbers are all examples of rational numbers: $\frac{3}{4}$, 5, $\frac{8}{3}$ and $1\frac{1}{2}$.

(T) 6. Zero is the identity element for addition of rational numbers.

(T) 7. The fractions $\frac{0}{a}$ and $\frac{0}{b}$ represent the same rational number if neither a nor b is zero.

(F) 8. If a and b are rational numbers, $\frac{a}{b}$ is always a rational number. (Note: 0 is a rational number; except for $b = 0$, the statement is true.)

(T) 9. A rational number multiplied by its reciprocal equals 1. (Note: If it is zero, it has no reciprocal.)

(T) 10. The symbol $\frac{24}{8}$ stands for a number which is both a whole number and a rational number.

(F) 11. The sum of two rational numbers whose fractions have equal numerators may be found by adding their denominators.

(T) 12. The product of zero and any rational number is zero.

(F) 13. If one fraction has a larger numerator than a second fraction, the number represented by the first fraction is always larger than the number represented by the second fraction.

(T) 14. Even if $a = 0$, $\frac{a}{7}$ is a rational number.

(F) 15. If two fractions have the same denominator, the numbers they represent are always equal.

(F) 16. The reciprocal of $\frac{13}{19}$ is $\frac{1}{\frac{19}{13}}$

(F) 17. The least common multiple of the denominators of $\frac{1}{2}$ and $\frac{5}{6}$ is 12.

(T) 18. In the division problem, $\frac{1}{2}$ divided by $\frac{1}{3}$, we are looking for a number which when multiplied by $\frac{1}{3}$ gives $\frac{1}{2}$.

(F) 19. In the division problem, $\frac{1}{2}$ divided by $\frac{1}{3}$, we are seeking a number which when multiplied by $\frac{1}{2}$ gives $\frac{1}{3}$.

(F) 20. The reciprocal of the reciprocal of 3 is $\frac{1}{3}$.

(F) 21. Even if b equals 0, $\frac{a}{b}$ is a rational number.

(F) 22. The sum: $\frac{a}{c} + \frac{b}{c}$ is equal to $\frac{a + b}{2c}$.

II. Multiple Choice

23. The sum: $\frac{r}{s} + \frac{t}{u}$ is equal to which of the following for all counting numbers r, s, t and u:

a. $\frac{r + t}{s + u}$

b. $\frac{r + t}{su}$

c. $\frac{rs + tu}{su}$

d. $\frac{st + ru}{su}$

e. None of these

23. d

24. Which of the following pairs of numbers are both divisible by the same number greater than one?

a. 7, 3. d. 5, 23.
b. 8, 9 e. None of these.
c. 7, 28

24. c.

25. The product: $\frac{x}{z} \cdot \frac{t}{k}$ is equal to which of the following if x, t, z and k are counting numbers:

a. x plus z plus t plus k .
b. $\frac{xk}{zt}$ d. (xt)(zk)
c. $\frac{xt}{zk}$ e. None of these.

25. c.

26. If $\frac{x}{a} = \frac{y}{b}$ and $a = 6$ and $b = 12$, then

a. $x = 2y$ d. $12x = 12y$
b. $y = 2x$ e. None of these.
c. $6x = y$

26. b.

27. If a and b are whole numbers $(b \neq 0)$ then $\frac{a}{b}$ is the number x for which

a. $x \cdot a = b$ d. $b + x = a$
b. $a \cdot b = x$ e. None of these.
c. $b \cdot x = a$

27. c.

III. General Questions

Perform the indicated operations.

28. $\frac{2}{3} + \frac{10}{15} =$ $\frac{4}{3}$

29. $\frac{1}{3} + \frac{2}{5} =$ $\frac{11}{15}$

30. $\frac{5}{9} + \frac{1}{9} =$ 5

31. $\frac{2}{7} \cdot \frac{3}{7} =$ $\frac{6}{49}$

32. $\frac{5}{7} - \frac{1}{7} =$ $\underline{\frac{4}{7}}$

33. $\frac{7}{11} - \frac{1}{9} =$ $\underline{\frac{52}{99}}$

Answers

34. $x = 0 \cdot (\frac{1}{3} + \frac{1}{2} + \frac{1}{6})$ $x = 0$

35. $x = \frac{1}{3} \div (\frac{1}{4} + \frac{1}{5})$ $x = \frac{20}{27}$

36. $\frac{x}{3} = \frac{6}{9}$ $x = 2$

37. $x = \frac{91 - 35}{7}$ $x = 8$

38. $x =$ product of $\frac{7}{10}$ and $\frac{10}{10}$ $x = \frac{7}{10}$

39. $x = (3 \cdot 5) \cdot (\frac{1}{5 \cdot 3})$ $x = 1$

40. $x = \frac{11}{24} \div \frac{11}{2}$ $x = \frac{1}{12}$

41. $x = \frac{2}{3} - \frac{3}{5}$ $x = \frac{1}{15}$

42. When Mr. Henry looked at the gauge on the dashboard of his car he saw that the gas tank was $\frac{3}{4}$ full. The next time he looked the tank was $\frac{1}{8}$ full. What part of a full tank of gas did Mr. Henry use? Ans. $\frac{5}{8}$

43. In a high school graduating class $\frac{5}{6}$ of the graduates plan to go to college. Of these, $\frac{3}{4}$ are boys. What part of the girls in the graduating class are planning to go to college? Ans. $\frac{5}{24}$

44. Of the total number of hits made by a major league baseball team, $\frac{3}{10}$ were doubles, $\frac{1}{20}$ were triples, and $\frac{1}{15}$ were home runs.

a. What part of the total hits were for extra bases?

b. What part of the total hits were singles?

Ans. a. $\frac{5}{12}$ b. $\frac{7}{12}$

45. At a party, pie was served with each pie cut into 6 equal pieces. Mary asked for $\frac{1}{2}$ of a portion. What part of a pie did Mary ask for? Ans. $\frac{1}{12}$

46. A steak weighs $\frac{7}{8}$ of a pound. If $\frac{1}{3}$ of the steak consists of bone and fat, what part of the steak can be eaten and how much does it weigh?

Ans. $\frac{2}{3}$ of the steak, which weighs $\frac{7}{12}$ lb.

Chapter 7

NON-METRIC GEOMETRY II

This chapter is a continuation of the work on non-metric geometry. The general discussion in the introduction to Chapter 4 applies in this chapter.

7-1. Segments.

These ideas are developed in this chapter:

(a) If A, B, and C are three points on a line, intuition tells us which point is between the other two.

(b) A segment is determined by any two points and is on the line containing those points.

(c) The two points which determine a segment are called endpoints of the segment.

(d) A segment is a set of points which consists of its endpoints and all points between them.

(e) If every member of a certain set is also a member of a second set, the first set is called a subset of the second.

(f) The union of two sets consists of all the elements of the two sets.

Bring out the idea that when we draw a sketch or a picture of a line, we draw a picture of one part of the line, and that this is properly, a line segment. However, we often represent a line by a part of a line (since we cannot do anything else). One should be careful to say that the sketch represents a line or segment as is appropriate.

Draw a representation of a line on the chalkboard and name two points of the line, A and B. Note that $\overline{AB}$ means points A and B and all points between them. Name other points on the line and various segments.

Review the idea of intersection of two sets. Exercises 7-1a will provide ample experiences for the students in applying the idea of intersection of sets in working with segments.

Answers to Exercises 7-1a

1. a. $\overline{AB}$ and $\overline{DE}$. For $\overline{AB}$ the endpoints are A and B. For $\overline{DE}$ the endpoints are D and E. The student might take $\overline{DC}$ and $\overline{CB}$ or $\overline{AC}$ and $\overline{CE}$.

b. $\overline{BE}$, $\overline{BD}$, $\overline{BC}$, or $\overline{BA}$. For $\overline{BE}$, endpoints are B and E. For $\overline{BC}$, endpoints are B and C. For $\overline{BA}$ endpoints are B and A. For $\overline{BD}$ endpoints are B and D.

c. $\overleftrightarrow{AD}$ is a line, unlimited in extent in both directions. $\overline{AD}$ is a segment, or portion of the line.

d. Point C

e. The empty set

2. a. $\overline{ZX}$, $\overline{ZV}$, $\overline{ZY}$, $\overline{ZW}$.

b. Point Y

c. Point V

d. The empty set

e. $\overline{XY}$

3. a. Any three of the following: $\overline{TK}$, $\overline{TD}$, $\overline{TF}$, $\overline{KD}$, $\overline{KF}$, $\overline{DF}$, $\overline{TF}$.

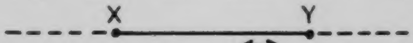
b. $\overline{EM}$

c. An unlimited number. Points K and D.

d. The empty set

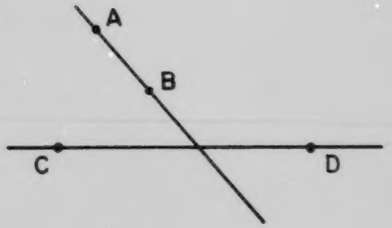
e. $\overline{AB}$, $\overline{AC}$, $\overline{AE}$, $\overline{AR}$, $\overline{AL}$, $\overline{BC}$, $\overline{BE}$, $\overline{BR}$, $\overline{BL}$, etc.

4. a. No.

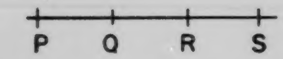


b. Yes, since the line $\overleftrightarrow{XY}$ extends beyond Y.

5\.



6\.



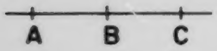
7. a. Yes, because A and B are on the same side of ℓ.
   b. No, because A and C are on opposite sides of ℓ.

The idea of subsets is very useful. In order to get practice with this concept, pupils might name a set of objects in the classroom and one of its subsets. For instance, the desks in one row are a subset of the set of all desks; the chalkboards on one side of the room are a subset of the set of all chalkboards. Then the pupils might name subsets of all triangles or of the set of all quadrilaterals.

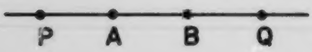
Develop the idea of union of two sets using the illustrations in the text. The idea of union is useful in dealing with segments and triangles. Exercises 7-1b will provide a variety of applications.

Answers to Exercises 7-1b

1. a. Yes b. Yes c. $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, $\overline{BD}$, $\overline{CD}$.
   $\overline{AD}$ is also a subset of $\overline{AD}$ because every set is a subset of itself.
2. a. Yes b. $\overrightarrow{CD}$, $\overrightarrow{CE}$, $\overrightarrow{DE}$
3. a. {a, b, c, d, e, i, o, u}; {a, e}
   b. {1, 3, 4, 5, 7, 9, 11, 16}; {1, 9}
   c. The set of whole numbers 1 through 12; {3, 4, 5, ...10}
   d. The set of men on the football team; the empty set.
4.



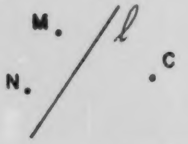
   a. Point B
   b. $\overline{BC}$
   c. $\overline{AC}$
   d. $\overline{AC}$
   e. $\overline{AC}$
   f. $\overline{AB}$
5. No.



   $\overline{PB}$, $\overline{PQ}$, $\overline{AQ}$, also contain A and B.

6. a. $\overline{PR}$
   b. $\overline{QS}$
   c. $\overline{PS}$
   d. $\overline{QS}$
   e. $\overline{PS}$
   f. $\overline{PS}$
   g. $\overline{PR}$

7. Yes, No. Since M and C are on opposite sides of ℓ, the line joining M and C must intersect ℓ.



8. a. Every element in the intersection of A and B is a member of both A and B and hence is a member of A.

   The union of A and B consists of every element in either A or B. Hence every element of A is in $A \cup B$.

   b. Let x be any member of A. Since A is contained in B, x is an element of B.
   But B is contained in C. Therefore x is a member of C.
   Since any member of A is a member of C, then A is contained in C.

9. These answers will vary. An example of sets of numbers is:

   (For 8a) Let $A = \{1, 2, 3, 4, 5, 6\}$
   Let $B = \{2, 4, 7\}$
   $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
   $A \cap B = \{2, 4\}$
   $A \cap B$ is contained in A.
   A is contained in $A \cup B$.

   (For 8b) Let $A = \{30, 60\}$
   Let $B = \{10, 20, 30, 40, 50, 60\}$
   Let $C = \{0, 10, 20, 30, 40, 50, 60, 70\}$
   A is contained in B
   B is contained in C
   A is contained in C.

10. a. B is a subset of A.
    b. C is a subset of A.
    c. $B \cap C$ is A

7-2. Separations.

These ideas are developed in this section:

(a) A plane separates space into two half-spaces.

(b) A line of a plane separates a plane into two half-planes.

(c) A point separates a line into two half-lines.

(d) A ray is the union of a half-line and the point which determines the half-line.

Use cardboard models to develop understanding of these ideas. This section gives an unusually good opportunity to emphasize relations among point, line, plane, and space. You can expect seventh grade students particularly to enjoy this section. It gives a certain structure to geometry on an intuitive basis.

Draw a number of lines on the chalkboard. Mark points on them and discuss half-lines, rays and endpoints. Discuss the intersection of two rays, two half-lines, and ray and half-line. If students inquire about whether a half-line has an endpoint the following explanation may be given. If a line is separated by a point then each half-line including the point of separation is a ray. If the point of separation is removed then we have two half-lines. We say that each of these half-lines has an endpoint, the endpoint of the corresponding ray. However, in the case of the half-line the endpoint is not a member of the set of points constituting the half-line. We speak of the graduates of a school even though the graduates are not physically present in the school.

Also, identify representations of half-spaces, produced by room-divider, walls in building; of half-planes, by line on paper, lines on wall, etc.; and of half-line by naming a particular point along the edge of a ruler.

Answers to Exercises 7-2a

1. a. A plane separates space into two half-spaces.
   b. A point separates a line into half-lines.
   c. A plane separates space into two half-spaces.
   d. A line of a plane separates the plane into two half-planes.
   e. A plane separates space into two half-spaces.
   f. A point separates a line into two half-lines. Should a student say that the 8-inch mark is a line, accept the interpretation that a line of a plane divides the plane into two half-planes.
   g. A line of a plane divides the plane into two half-planes.

2. a. True
   b. False
   c. True
   d. True
   e. False
   f. False

3. a. Yes. A line of a plane separates the plane into two half-planes.
   b. No. Since $\overline{PQ}$ is a segment it is limited in length. It cannot separate the plane.

4. Yes. If lines k and m are extended without limit in both directions every point on m will be on the P-side of k.

5. Yes.

*6. a. Yes, by choosing the two half-lines on the same line so that they overlap, but not by choosing two half-lines having the same endpoint since half-lines do not contain their endpoints.
   b. Yes, by choosing two overlapping half-planes lying in the same plane with their edges parallel, but not by choosing two half-planes with a common edge since half-planes do not contain their edges.

*7. No. Yes.

Answers to Exercises 7-2b

1. a. The line $\overleftrightarrow{RS}$. It can be extended without limit in both directions. Line $\overleftrightarrow{RS}$ should be thought of as a set of points.
   b. The segment $\overline{RS}$. This is the set of points on line $\overleftrightarrow{RS}$ between points R and S and including the endpoints R and S.
   c. The ray $\overrightarrow{RS}$. The ray has the endpoint R and can be extended without limit in the direction of S. A ray should also be thought of as a set of points.
2. a. $\overline{PK}$
   b. Point L
   c. $\overleftrightarrow{PK}$. Note to teacher--This union of two rays does not result in an angle since the rays are on one straight line.
3. a. $\overline{AB}$; b. $\overrightarrow{BA}$; c. $\overline{AB}$; d. $\overleftrightarrow{AC}$; e. $\overrightarrow{CB}$ or $\overrightarrow{CA}$
4. There are several correct answers. One set follows.
   a. $\overrightarrow{BA} \cup \overrightarrow{BC}$
   b. $\overrightarrow{BA} \cup \overrightarrow{CD}$
   c. $\overrightarrow{BC} \cup \overrightarrow{CD}$
   d. $\overrightarrow{BA} \cap \overrightarrow{BC}$
   e. $\overrightarrow{BA} \cap \overrightarrow{CD}$.

7-3. Angles and Triangles.

In this section students should learn that:

(a) An angle is a set of points consisting of two rays not both on the same straight line and having an endpoint in common.

(b) An angle separates the plane containing it.

(c) A triangle is the union of three sets, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ where A, B, and C are any points not on the same line.

(d) A triangle determines its angles but does not contain its angles.

Illustrate the idea of angle as two rays with the same endpoint. Use colored chalk to show interior and exterior. Note how an angle is named.

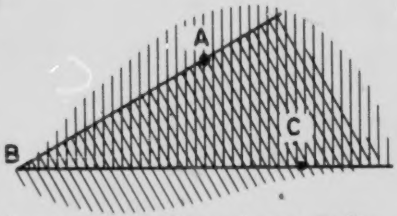
In developing the idea of triangle, put three points on the board and note them as endpoints of 3 line segments, $\overline{AB}$, $\overline{BC}$, $\overline{AC}$. Note the set of points in each segment and that a triangle is the union of these three sets. Use colored chalk to show interior and exterior. Emphasize the set of points of the interior, the exterior and that of the triangle.

Again, students may be interested in drawing angles, triangles, shading, etc. (This is not perspective drawing.) Drawing is a good way to show the students concrete representation of abstract ideas. It also helps to develop imagination and to see relationships.

In discussing the angles of a triangle bring out the idea that although people often talk about angles of a triangle, it is a short way of saying that they are the angles determined by the triangle. For example, a city "has" suburbs, but the suburbs are not part of the city.

Answers to Exercises 7-3a

5. The crosshatched section is the interior of the angle.



Answers to Exercises 7-3b

1. a. $\angle XZY$, $\angle YZX$
   b. Point Z
   c. $\overrightarrow{ZX}$ and $\overrightarrow{ZY}$
2. c. The interior of $\angle ABC$
3. a. $\angle TVW$
   b. Point V
   c. Point V

Answers to Exercises 7-3c

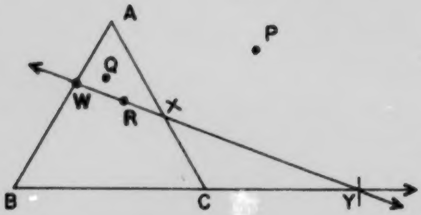
1. a. The interior of ∠ ABC
   b. The interior of Δ ABC

2. a. Yes. They have different vertices.
   b. Yes
   c. The lines containing the rays determine a plane. (Property 3)

3. a. The point A
   b. No. Half-lines or rays would extend beyond the triangle.
   c. AB
   d. AB
   e. ∠ BAC

4. a. Points X and W
   b. Δ ABC, Δ AWX, Δ XCY, Δ YBW
   c. No
   d. A, B, C, W, and Y;
      A, W, and B are in exterior of Δ XCY
      Y is in exterior of Δ ABC
      A is in exterior of Δ BWY
      B, C, and Y are in exterior of Δ AWX.
   e. Point B, Point A

5.



   c. R can be on $\overline{WX}$

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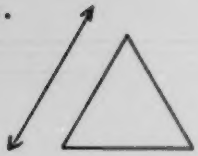
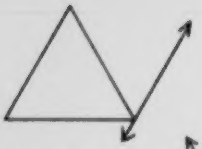
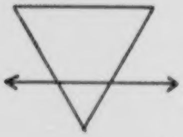
6. a.



b.

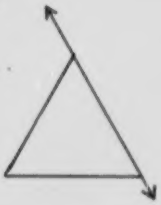


c.

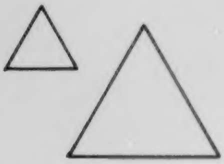


d. Not possible

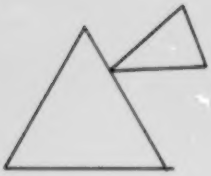
e.



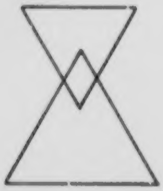
7. a.



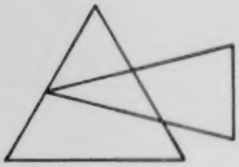
b.



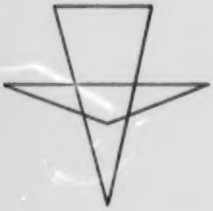
c.



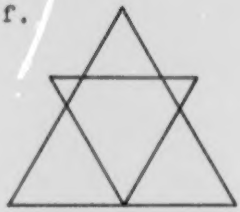
d.



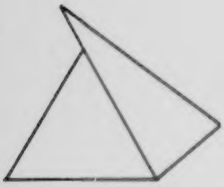
e.



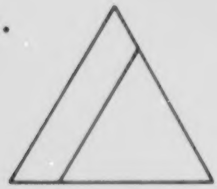
f.



g.



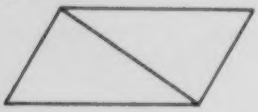
h.



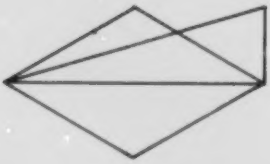
i. The two triangles are such that one is superimposed upon the other.

8. a. The points A and C
   b. $\overline{AB}$
   c. The points A and B
   d. The point B
   e. $\overline{BC}$
   f. $\overline{BC}$
   g. The union of $\overline{AB}$ and $\overline{BC}$
   h. $\angle$ ACB

*9. a. No



Yes



10. BRAINBUSTER:
   * a. Yes
   b. It may or may not, depending upon choice of P and Q.
   c. Yes
   d. No

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*7-4. One-to-One Correspondence. (Optional)

Note that this section is considered optional. The ideas of one-to-one correspondence which were learned in Chapter 3 are extended to geometry, and correspondence between points and lines is investigated.

The idea of one-to-one correspondence is fundamental in counting.

One-to-one correspondence in geometry can be established

(1) Between a certain set of lines and a certain set of points

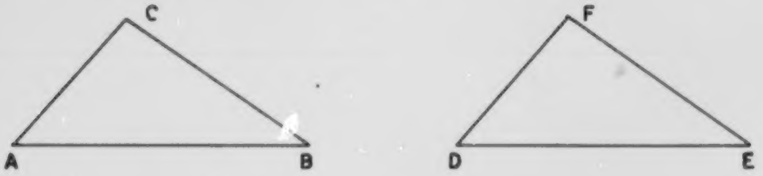
(2) Between the set of points of one segment and the set of points of another segment.

Review the idea of one-to-one correspondence and the necessary condition that for each element in set A there corresponds an element in set B and for each element in set B there corresponds an element in set A. For example, if there are 5 chairs and 5 people, for each chair there is a person and for each person there is a chair.

This idea, while elementary, is sometimes hard to grasp. One-to-one correspondences between finite sets(sets having a specific number of elements as in the illustration above) are easy to observe if they exist. Encourage pupils to suggest examples that they observe.

Background Material for Teacher---While the following discussion is not directly related to the material in this chapter it presents a point of view that is useful in more advanced grades.

We are sometimes interested in a particular one of the one-to-one correspondences. For the two congruent triangles below we are interested in matching A with D, B with E, and C with F. It is on such basis that we get the congruence. If we were to match A with F, B with D, and C with E we would not be noting the congruence.



For infinite sets H and K we may be interested in two aspects:

(1) Is there any one-to-one correspondence between H and K?

(2) Is there a "nice" or "natural" one-to-one correspondence?

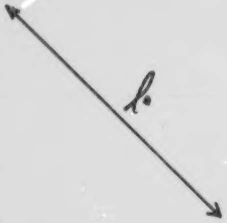
In the examples in Section 7-4, we not only show that there is some one-to-one correspondence but that there is a "natural" or "nice" one. There also would be a great many that are not "natural" or "nice."

To establish a one-to-one correspondence we need (1) a complete matching scheme, and (2) in this particular device it must be true that for any element of either set there corresponds a unique element of the other set. It is implied by what we say that if a corresponds to b, then b corresponds to a.

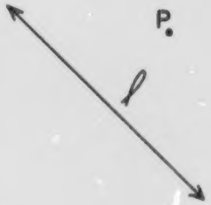
In effect, to establish a one-to-one correspondence we must have a way of "tying" each element of either set to a particular element of the other. And the "string" we use for tying a to b also ties b to a.

Answers to Exercises 7-4a

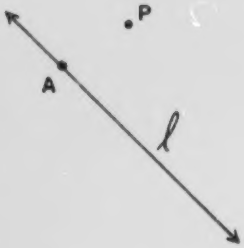
1.



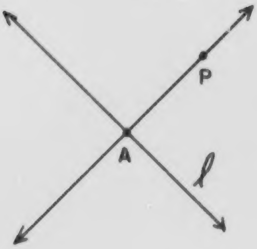
2.



3.

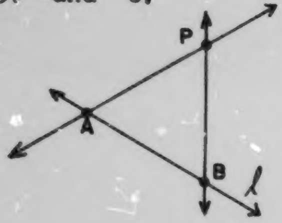


4.

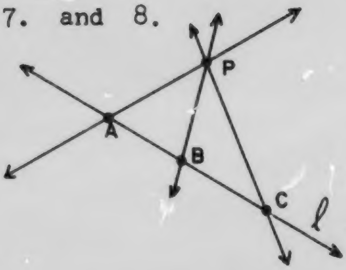


a. Yes
b. One

5. and 6.



7. and 8.



6. a. Yes
   b. One

8. a. Yes
   b. One

9. a. Yes
   b. Yes

10. a. 3
    b. 3
    c. Yes
    d. Yes
    e. Yes
    f. Yes
    g. Point, point, line.

Answers to Exercises 7-4b

1. a. Yes, provided that each pupil has a desk, and each desk is assigned to a pupil.
   b. No. There may be more pupils than desks. There may be more desks than pupils. Some pupils may be absent on a given day.
2. a. Yes
   b. corresponding, right hand
3. The members of one team can be matched one-to-one with the members of the other team.
4. Each point has an opposite side and each side has an opposite point, as follows:

   A and $\overline{BC}$, B and $\overline{AC}$, C and $\overline{AB}$
5. a. one
   b. one point
   c. Yes

6. a. To each point on $\overline{DE}$ there corresponds an element of K and to each element of K there corresponds a point on $\overline{DE}$.

   b. Similar analysis.

   c. To each point on $\overline{DE}$ there corresponds a point on $\overline{XZ}$ and to each point on $\overline{XZ}$ there corresponds a point on $\overline{DE}$. These corresponding points are determined by K, the set of all the rays through Y which do not contain points in the exterior of $\angle XYZ$.

*7. Each even whole number is matched with the odd number that is its successor.

$$\begin{array}{cccccc} 0 & 2 & 4 & 6 & 8 & \ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 3 & 5 & 7 & 9 & \ldots\ldots \end{array}$$

*8. Each whole number is matched with the whole number which is twice its value.

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \ldots\ldots \\ 0 & 2 & 4 & 6 & 8 & \end{array}$$

7-5. Simple Closed Curves.

In this section these ideas are presented:

(a) Broken-line figures such as those we see in statistical graphs, triangles, rectangles, as well as circles and figure eights are curves.

(b) A simple closed curve in the plane separates the plane into two sets---the points in the interior of the curve and the points in the exterior of the curve. The curve itself is contained in neither set.

(c) The curve is called the boundary of the interior (or the exterior).

(d) The interior of a simple closed curve is called a region.

(e) The interior of a simple closed curve together with its boundary is called a closed region.

(f) If a point A is in the interior of a curve and a point B is in the exterior of the curve, then the intersection of $\overline{AB}$ and the curve contains at least one element.

Draw some curves on the chalkboard, bringing out the idea that we call them "curves" and that a segment is just one kind of curve.

Note that a simple closed curve separates a plane into two sets and that the curve itself is the boundary of the two sets. Also, that any quadrilateral, parallelogram or rectangle is a simple closed curve. Identify some of the many curves which are suggested in the room, such as boundary of chalkboard, total boundary of floor surface, etc.

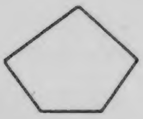
Students enjoy drawing elaborate curves which may still be classified as simple closed curves. Encourage their drawing a few simple closed curves for a bulletin board exhibit.

Answers to Exercises 7-5

1. a.

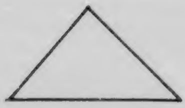
Any quadrilateral

b.



Any pentagon

c.



Any triangle

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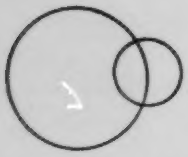
2.



3. d. The intersection of the exterior of C_2 and the interior of C_1.

   e. The interior of C_2 and the exterior of C_1.

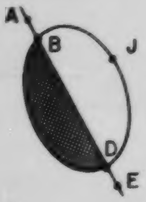
4. a. Yes

   b. Yes

   c. No. It contains 2 intersections.

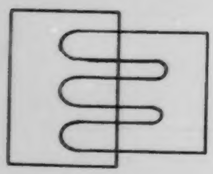
5. a. B and D

   b.



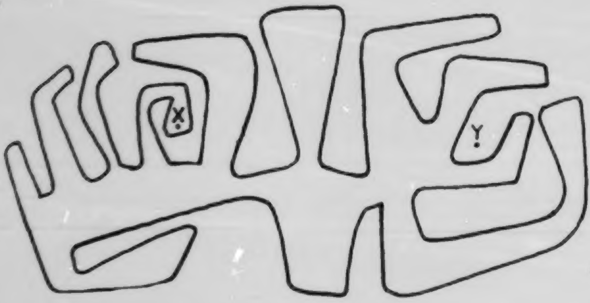
   c. $\overrightarrow{BA}$ and $\overrightarrow{DE}$

6.



7. a. Any simple closed curve with either X or Y in the interior.

   b.



Note that X and Y are separated.

7-7. Chapter Review.

Answers to Exercises 7-7

1. a. $\overrightarrow{SR}$, $\overrightarrow{ST}$ or $\overrightarrow{SU}$
   b. $\overrightarrow{SR}$ and $\overrightarrow{ST}$, etc.
   c. $\overrightarrow{SR} \cup \overrightarrow{TU}$, etc.
   d. $\overrightarrow{ST} \cup \overrightarrow{TU}$, etc.
   e. $\overrightarrow{SR} \cap \overrightarrow{ST}$, etc.
   f. $\overrightarrow{SR} \cap \overrightarrow{TU}$, etc.

2. $\overline{RS}$, $\overline{RT}$,
   $\overline{RU}$, $\overline{ST}$, $\overline{SU}$, $\overline{TU}$,
   $\overrightarrow{RU}$, etc.

3. a. $\overleftrightarrow{AD}$ is the line $\overleftrightarrow{AD}$. It extends without limit in both directions. $\overline{AD}$ is the segment $\overline{AD}$. It does not extend beyond its endpoints.
   b. $\overline{AB}$ is the segment $\overline{AB}$. It does not extend beyond its endpoints. $\overrightarrow{AB}$ is the ray $\overrightarrow{AB}$. It has one endpoint, A, and it extends without limit in the direction indicated by starting with A and proceeding through B.
   c. Point D

4. a. Point A
   b. $\overline{BD}$
   c. Point C
   d. The empty set

5. a. $\overline{PR}$ and $\overline{QS}$, etc.
   b. $\overline{PQ}$ and $\overline{QR}$, etc.
   c. $\overline{PQ}$ and $\overline{RS}$
   d. $\overline{PQ}$ and $\overline{RS}$

6. a. $R \cap S = \{7, 10\}$
   b. $R \cup S = \{1, 3, 4, 5, 7, 9, 10\}$

7. a. G or B
   b. A or F
   c. E
   d. The empty set
   e. $\overline{FG}$
   f. $\angle$ BEG

8. a. It divides space into two half-spaces.
   b. It divides the plane of the basketball court into two half-planes.
   c. It divides space into two half-spaces.
   d. Point P divides both lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ into half-lines.

9. a. $\angle ABC$
   b. A
   c. Segments
   d. $\overline{AB}$
   e. $\overline{AB}$

10. a. Point E.
    b. $\overline{DC}$
    c. $\overline{EC}$
    d. $\angle AED$
    e. The empty set

11. a. $\overleftrightarrow{AB}$ or $\overleftrightarrow{PR}$
    b. B
    c. $\angle BQS$
    d. Q and S
    e. Q

12. d. The exterior of $C_2 \cap$ the interior of C_1.

13. We can match each state with its state capital and we can match each state capital with its state.

14. a. A
    b. G and F
    c. A, D, or E
    d. $\angle CAF$
    e. $\overline{AC}$

15. Each line through Z will cut segments $\overline{XY}$ and $\overline{WV}$ in matching points.

7-8. Cumulative Review.

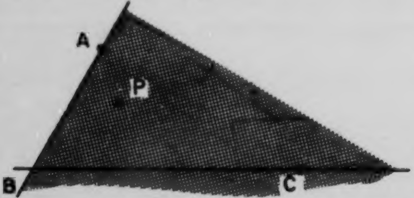
Answers to Exercises 7-8

1. False
2. True
3. True
4. False
5. True
6. True
7. False
8. True
9. Eight
10. a. 8
    b. any whole number
    c. 0
11. a. J
    b. the empty set
    c. the empty set
    d. $\overleftrightarrow{JK}$
    e. H
12. a. 2, 3, 4, 5, 6, 10
    b. 3, 5
13. a. $\frac{5}{7}$
    b. $\frac{15}{14}$
    c. $\frac{1}{15}$
    d. $\frac{3}{14}$
14. a. $\overline{EF}$
    b. point E
    c. If 2 different lines intersect, one and only one plane contains both lines.
    d. $\overleftrightarrow{EG}$ and $\overleftrightarrow{GF}$ are in exactly one plane. E is a point on $\overleftrightarrow{EG}$ and F is a point on $\overleftrightarrow{GF}$. Therefore, E and F are in plane EGF and $\overleftrightarrow{EF}$ is in plane EGF.

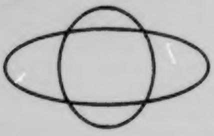
Sample Test Questions for Chapter 7

Teachers should construct their own tests, using carefully selected items from those given here and from their own. There are too many questions here for one test. Careful attention should be given to difficulty of items and time required to complete the test.

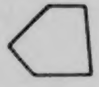
1. Draw $\angle ABC$. Label P a point in the interior. Shade the P side of line $\overleftrightarrow{AB}$.



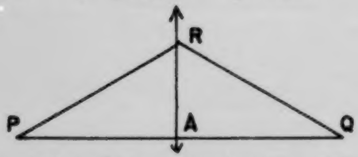
2. Draw two simple closed curves whose intersection is a set of exactly four points.



3. Draw a simple closed curve which is the union of five segments, no two on the same line.

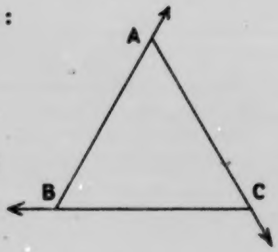


4. Draw $\triangle PQR$. Label a point A between P and Q. Draw the line $\overleftrightarrow{AR}$. List all triangles represented in your figure. ($\triangle PQR$, $\triangle PAR$, $\triangle RAQ$)

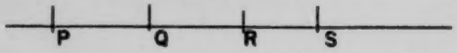


5. Each point of a line separates the line into _____ _____.
Each line of a plane separates the plane into _____ _____.
A plane separates space into _____ _____.
(two half-lines, two half-planes, two half-spaces).

6. Consider the figure at the right:
   a. List 3 rays represented
   ($\overrightarrow{BA}$, $\overrightarrow{AC}$, $\overrightarrow{CB}$)
   b. What is $\overrightarrow{AC} \cap \triangle ABC$? ($\overline{AC}$)
   c. What is $\overline{AB} \cap \overrightarrow{AC}$? (Point A)



7. Draw a horizontal line. Label four points on it P, Q, R and S in that order from left to right.



a. Name two segments whose intersection is one point. ($\overline{PQ}$ and $\overline{QR}$, etc.)

b. Name two rays whose union is the line. ($\overrightarrow{PQ}$ and $\overrightarrow{QP}$, etc.)

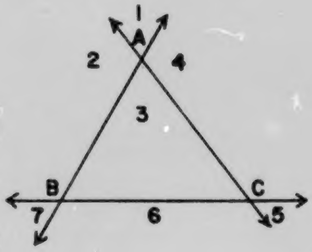
c. Name two segments whose intersection is a segment. ($\overline{PR}$ and $\overline{QS}$, etc.)

d. Name two segments whose union is not a segment. ($\overline{PQ}$ and $\overline{RS}$, etc.)

8. Label A, B, and C three points not all on the same line. Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.

a. Into how many regions does the union of the segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ separate the plane? (2)

b. Into how many regions does the union of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ separate the plane? (7)



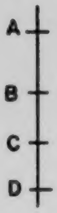
9. Draw a vertical line on your paper. Label points A, B, C and D in that order from top to bottom.

a. What is $\overline{AC} \cap \overline{BD}$? ($\overline{BC}$)

b. What is the $\overrightarrow{BA} \cap \overrightarrow{CD}$? (The empty set)

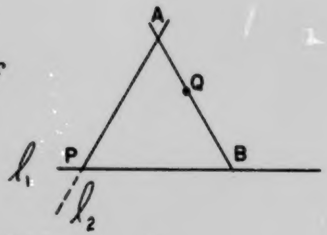
c. What is the union of $\overline{AB}$ and $\overrightarrow{BC}$? ($\overrightarrow{AB}$)

d. What is the union of $\overline{AD}$ and $\overline{BC}$? ($\overline{AD}$)



10. In the figure, show how a set of rays from P may be used to establish a one-to-one correspondence between $\overline{AC}$ and $\overline{BD}$. (Rays from P will cut $\overline{AC}$ and $\overline{BD}$ in sets of points which may be shown to correspond.)

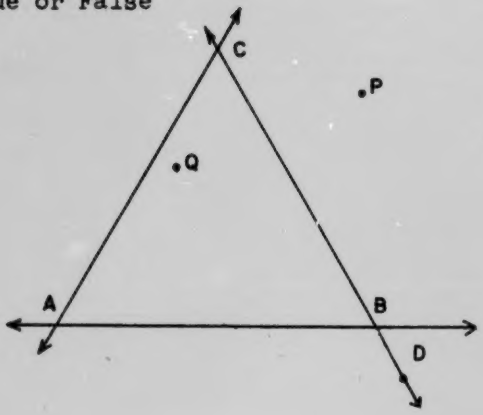
11. To the left of an item in the left-hand column place the letter of a corresponding item from the right-hand column:



(g) 1. the union of $\overrightarrow{PA}$ and $\overrightarrow{PB}$
(f) 2. $\ell_1 \cap \ell_2$
(c) 3. segment $\overline{AQ}$
(d) 4. point in interior of $\angle APB$
(e) 5. $\overleftrightarrow{PB}$
(a) 6. ray on ℓ_2
(b) 7. the union of $\overline{AQ}$ and $\overline{QB}$
(i) 8. $\overline{QB} \cap \ell_1$
(h) 9. $\overline{AQ} \cap \ell_1$
(j) 10. the union of ℓ_2 and $\overline{AP}$

a. $\overrightarrow{AP}$
b. $\overline{AB}$
c. $\overline{AQ} \cap \overline{AB}$
d. Q
e. ℓ_1
f. P
g. $\angle APB$
h. the empty set
i. B
j. ℓ_2

12. True or False



In the corresponding blank to the left of each of the following statements indicate if it is true or false.

(True) 1. Point Q is on the C-side of $\overleftrightarrow{AB}$.
(False) 2. $\overleftrightarrow{AB}$ is the intersection of two half-planes.

(False) 3. The union of $\overrightarrow{CD}$ and $\overrightarrow{CA}$ is $\angle ABC$.

(False) 4. Point B separates $\overrightarrow{CD}$ into two segments $\overline{CB}$ and $\overline{BD}$.

(False) 5. Point B separates $\overline{CD}$ into two half-lines.

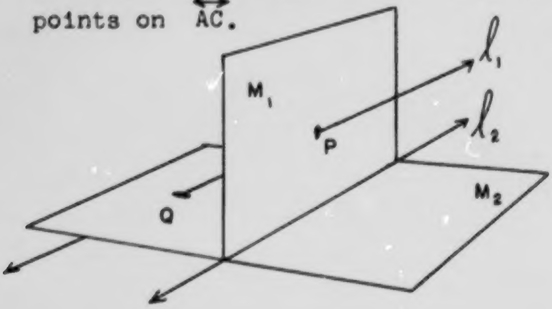
(True) 6. Point Q is in the interior of $\angle ACB$.

(False) 7. Point Q is in the interior of $\angle DBA$.

(True) 8. The interior of angle ABC is the intersection of a half-plane containing C and a half-plane containing A.

(True) 9. Triangle ABC is a simple closed curve.

(True) 10. A one-to-one correspondence may be established between (a) the set of lines intersecting $\overleftrightarrow{AC}$ and containing B and (b) the set of points on $\overleftrightarrow{AC}$.



13. Multiple Choice

1. ℓ_1 and ℓ_2
    a. intersect in the empty set
    b. are parallel
    c. intersect in one point
    d. are in the same plane

1 (a)

2. Segment $\overline{QP}$
    a. lies in plane M_2
    b. lies in plane M_1
    c. connects an element of M_1 with an element of M_2
    d. none of these

2 (c)

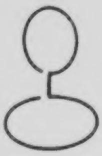
3. a. ℓ_2 separates M_1 into two half-planes
   b. $\ell_2 \cap M_2$ is the empty set
   c. $\ell_2 \cap M_1$ is the empty set
   d. ℓ_2 separates space 3 (a)

4. a. $M_2 \cap \ell_1 = P$
   b. $M_2 \cap M_1 = \ell_2$
   c. M_2 does not extend endlessly as does ℓ_2
   d. M_2 contains all the points of ℓ_1 4 (b)

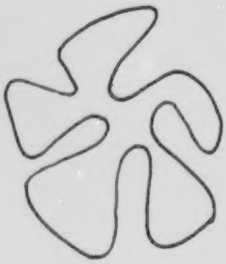
5. Point P:
   a. is in the intersection of M_1 and M_2
   b. lies in a half-plane whose boundary is ℓ_2
   c. is not an element of $\overline{PQ}$
   d. is not an element of ℓ_1 5 (b)

14. Are each of the following simple closed curves? Give a reason for your answer.

a. (No. It cannot be drawn so that it starts and stops at the same point without touching some point, other than the starting point, more than once.)



b. (Yes. It can be drawn so that it starts and stops at the same point without touching some point, other than the starting point, more than once.)



Chapter 8

RATIONAL NUMBERS AND THE NUMBER LINE

As the title indicates, this chapter deals primarily with rational numbers on the number line. A few other topics dealing with rational numbers have also been included in this chapter.

The importance of the idea of representing numbers as points on a line can hardly be over-emphasized. This idea provides the basis for all graphing and for coordinate geometry and is constantly used in mathematical analysis. Further, it provides the motivation for the order relations between numbers and for the invention of the real numbers. Also, this geometrical treatment of numbers should help to clarify the meaning of the elementary arithmetical operations of addition and multiplication. Most persons understand ideas better when they can be demonstrated geometrically.

8-1. The Number Line.

The placement of the whole numbers on the number line was introduced in Chapter 3. Now this topic is presented in more detail and the operations of addition, subtraction, multiplication, and division are treated geometrically.

Although we begin by placing arrows on both ends of the number line to indicate lines of infinite length, we later drop these for convenience.

Addition is vector addition in one dimension. Subtraction is presented as the inverse operation of addition so that the same diagram illustrates $2 + 3 = 5$ and $5 - 2 = 3$. This idea of inverse operation was presented in Chapter 3. There are other schemes for representing subtraction on the number line which are perfectly correct, but we have chosen this method of presentation which is consistent with the approach used when the negative rationals are studied in Chapter 17. The length and direction of the arrows are significant.

The method of adding is developed step by step in the text. Although it is somewhat awkward to transfer the arrow as is done for addition, this method not only gives the diagram for subtraction, but also can be generalized when negative numbers are introduced. After the student understands the procedure, the actual step of transferring the arrow to the origin can be done mentally, and need not appear on the diagram.

The student should go through each step carefully in doing the problems. It will save time if the students are given duplicated copies of number lines on which to do the exercises.

Answers to Exercises 8-1a

1. $3 + 4 = 7$ and $7 - 3 = 4$
2. $2 + 7 = 9$ and $9 - 2 = 7$
3. $4 + 4 = 8$ and $8 - 4 = 4$
4. $3 + 5 = 8$ and $8 - 3 = 5$

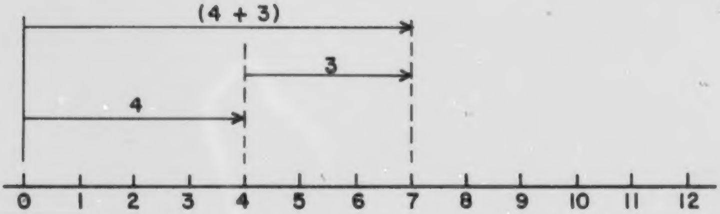
*5. $0 + 7 = 7$ and $7 - 0 = 7$

Multiplication and division on the number line are presented as inverse operations and the diagram for $3 \cdot 4 = 12$ is also the diagram for $12 \div 3 = 4$.

The only division problems considered in this section are those in which the quotients are whole numbers. In the next section the same method is applied in cases where the quotients are not whole numbers. As a result we obtain a method for locating arbitrary rational numbers on the number line.

Answers to Exercises 8-1b

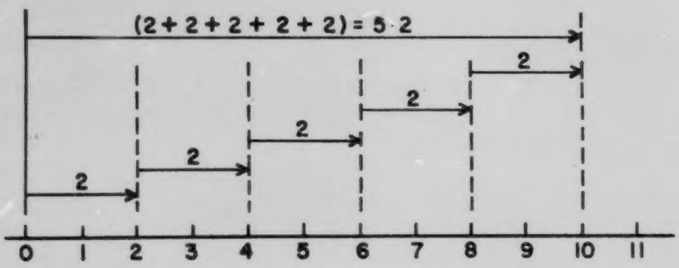
1. a.



By this time the student should be able to see the diagram in this form without transferring the arrow for 3. When we discuss the problem for addition of rational numbers we shall have to transfer the arrow since it would not be possible to locate the rational point without starting at 0.

b. 7 - 4. Same diagram as above.

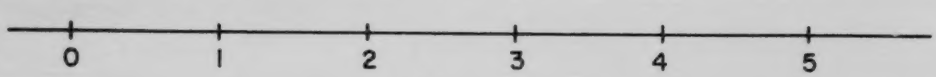
2. a. 5 · 2



b. 10 ÷ 5
Same diagram but labeled 10 and 10 + 5 instead of 5 · 2 and 2.

8-2. Locating Rational Numbers on the Number Line.

The process of finding rational numbers by folding ribbon parallels the motivation for the invention of the rational numbers in Section 6-2. A much more satisfactory construction exists. We present it here. Consider the number line:



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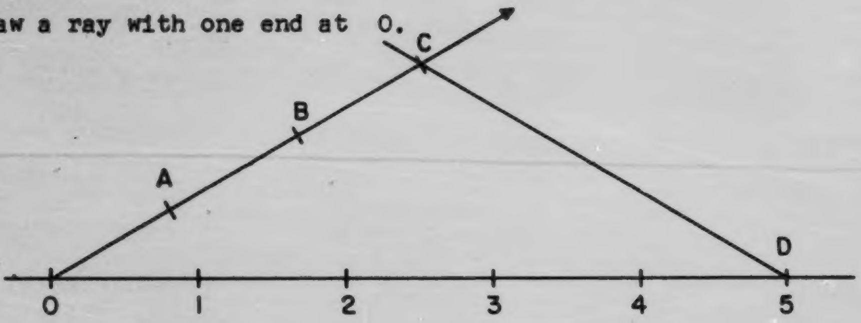
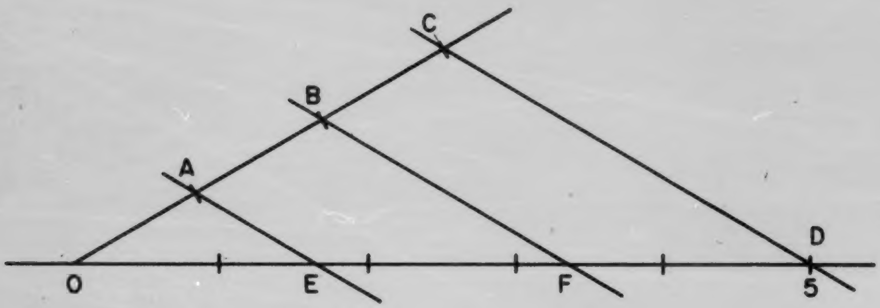
Draw a ray with one end at O.



On this ray mark off three equal segments as indicated above. Now draw the line through C and 5. Next draw lines through A and B parallel to CD.



And now the lengths OE, EF and FD are in the same proportion as OA, AB and BC. Since OA, AB and BC are equal, so are OE, EF and FD. The points E and F divide the segment from 0 to 5 into three equal parts and therefore the point labeled E is in fact $\frac{5}{3}$.

The above demonstration obviously requires a much greater knowledge of geometry than the student is equipped with at this time.

It should be noted that we first present $\frac{5}{3}$ as "one of three equal parts of 5" rather than "5 of three equal parts of one." This is in accord with the algebraic definition of $\frac{5}{3}$ given in Chapter 6 (i.e., $\frac{5}{3}$ is the result of dividing 5 by 3).

Answers to Exercises 8-2

1. a. $x = \frac{8}{3}$ b. $3x = 8$

2. a. $x = \frac{6}{4}$ c. $6 \div 4 = \frac{6}{4}$

   b. $4 \cdot \frac{6}{4} = 6$ d. $4x = 6$

3. a. $3 \cdot \frac{5}{3} = 5$ and $5 \div 3 = \frac{5}{3}$

   b. $4 \cdot \frac{7}{4} = 7$ and $7 \div 4 = \frac{7}{4}$

8-3. Comparing Rational Numbers.

The principal reason for representing numbers as points on a line lies in the fact that this representation preserves the order of the numbers. That is, if a is to the left of b then a is less than b and conversely. It is hoped that this section will impress this fact on the student. In addition to this geometric method for comparing numbers, the section supplies an algebraic method for comparing numbers.

In this section there is a diagram showing points labeled with numerals in several different ways. It might be well to emphasize here that the points on the number line correspond to numbers and not just to the numerals. That is, although a rational number may be represented by many different numerals, there is only one point on the number line corresponding to this number. In fact, there is nothing wrong with taking the point of view that the points on the number line are the numbers. Remember that we have never said anything concerning what numbers are. We have only discussed the properties of numbers or the ways in which numbers behave. But the numbers themselves are abstract entities; we have never decided what they are. In fact, we do not need to know what they are; we only need to know their properties. This is the idea behind the axiomatic method in mathematics.

Since we have not said what the numbers are, we are still at liberty to decide what they are. If we find it convenient, we may decide that the rational numbers are points on a certain line, of which we draw pictures in this chapter. This was the point of view taken in writing this chapter (although it was not specifically mentioned in the text nor need it be mentioned to the students).

The primary purpose of this chapter is to establish the geometrical interpretation of $>$ and $<$. There has been no effort to develop a comprehensive theory of inequalities.

Answers to Exercises 8-3a

1. 1
2. 2
3. $\frac{12}{6} > \frac{6}{6}$
4. $\frac{12}{6}$
5. $\frac{2}{6}$
6. a. $\frac{2}{6}$ b. $\frac{1}{6} < \frac{2}{6}$
7. a. $\frac{9}{6} < \frac{10}{6}$ b. $\frac{10}{6}$
8. If two fractions have the same denominator, the one with the largest numerator represents the greater number.
9. If two different numbers are located on the number line, the one farther to the right is the greater.

Answers to Exercises 8-3b

1. a. $\frac{9}{6} < \frac{10}{6}$
   b. $\frac{11}{3} < \frac{13}{3}$
   c. $\frac{6}{5} > \frac{4}{5}$
   d. $\frac{8}{7} < \frac{11}{7}$
   e. $\frac{3}{12} < \frac{4}{12}$
   f. $\frac{6}{9} > \frac{4}{9}$
   g. $\frac{4}{19} < \frac{22}{19}$
   h. $\frac{13}{24} < \frac{14}{24}$
   i. $\frac{11}{2} < \frac{12}{2}$
2. a. $q > r$ since q is to the right of r on the number line.
   b. $p < s$ since p is to the left of s on the number line.

c. $t < q$ since t is to the left of q on the number line.

d. $s > u$ since s is to the right of u on the number line.

e. $s < q$ since s is to the left of q on the number line.

f. $p < t$ since p is to the left of t on the number line.

g. $u > r$ since u is to the right of r on the number line.

h. $r < t$ since r is to the left of t on the number line.

i. $u > p$ since u is to the right of p on the number line.

Answers to Exercises 8-3c

1. a. $\frac{3}{4} > \frac{5}{9}$

$\frac{3}{4} \cdot \frac{9}{9} = \frac{27}{36}$

$\frac{5}{9} \cdot \frac{4}{4} = \frac{20}{36}$

$\frac{27}{36} > \frac{20}{36}$ therefore $\frac{3}{4} > \frac{5}{9}$

b. $\frac{4}{2} < \frac{7}{3}$

$\frac{4}{2} \cdot \frac{3}{3} = \frac{12}{6}$

$\frac{7}{3} \cdot \frac{2}{2} = \frac{14}{6}$

$\frac{12}{6} < \frac{14}{6}$, therefore $\frac{4}{2} < \frac{7}{3}$

c. $\frac{21}{22} > \frac{10}{11}$

$\frac{10}{11} \cdot \frac{2}{2} = \frac{20}{22}$

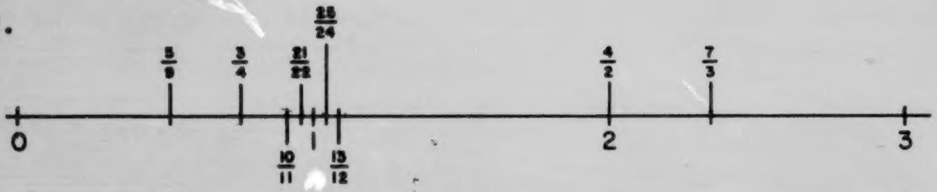
$\frac{21}{22} > \frac{20}{22}$, therefore $\frac{21}{22} > \frac{10}{11}$

d. $\frac{13}{12} > \frac{25}{24}$

$\frac{13}{12} \cdot \frac{2}{2} = \frac{26}{24}$

$\frac{26}{24} > \frac{25}{24}$, therefore $\frac{13}{12} > \frac{25}{24}$

2.



3. Store B since $\frac{3}{4} < \frac{7}{8}$.

4. The state since $\frac{4}{15} < \frac{3}{10}$.

The Comparison Property is given in this section. By deriving it from the problem of comparing only the products $a \cdot d$ and $b \cdot c$ for the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, the rule for equality is also derived. This gives the trichotomy statement for rational numbers although this property is only tacitly assumed for counting numbers throughout the text. There is no need to use the name, "trichotomy", with students.

Students may find it difficult to apply the Comparison Property. When they compare two rational numbers they will probably find it easier to express the numbers as fractions having a common denominator. Since we will make extensive use of the Comparison Property (in the section on equivalent fractions, when working with infinite decimals, and throughout Chapter 10 on ratios and proportion) it is important to do Exercises 8-3d in a very formal manner. The use of correct form now will make subsequent problems very easy. Some teachers may recognize the Comparison Property as the "cross product" rule; it might be helpful to some students to see that

$a \cdot d$ is $\frac{a}{b} \searrow \frac{c}{d}$ this product

and $b \cdot c$ is $\frac{a}{b} \nearrow \frac{c}{d}$ this product.

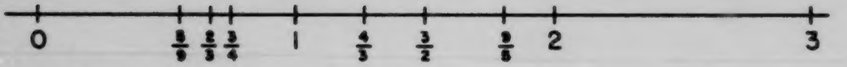
In some classes it is inadvisable to mention this for pupils apply it to $\frac{a}{b} + \frac{c}{d}$ as well.

The converse of each part of the Comparison Property is true, but we state only the converse for equality, since it is used extensively in the chapter on ratios and proportion.

Answers to Exercises 8-3d

1. $\frac{9}{11} < \frac{16}{19}$
2. $\frac{15}{4} > \frac{11}{3}$
3. $\frac{10}{15} = \frac{8}{12}$
4. $\frac{3}{4} < \frac{10}{13}$
5. $\frac{15}{4} > \frac{18}{5}$
6. $\frac{3}{7} > \frac{5}{12}$
7. $\frac{14}{19} > \frac{2}{3}$
8. $\frac{4}{7} < \frac{3}{5}$
9. $\frac{3}{17} < \frac{1}{5}$
10. $\frac{3}{8} > \frac{1}{3}$
11. $\frac{4}{3} > \frac{5}{4}$
12. $\frac{4}{6} = \frac{6}{9}$
13. $\frac{11}{8} > \frac{4}{3}$
14. $\frac{7}{4} > \frac{5}{3}$
15. $\frac{19}{11} < \frac{7}{4}$
16. $\frac{14}{5} < \frac{31}{11}$
17. $\frac{6}{8} = \frac{15}{20}$
18. $\frac{2}{9} > \frac{1}{5}$
19. $\frac{14}{6} = \frac{35}{15}$
20. $\frac{9}{7} > \frac{5}{4}$
21. $\frac{12}{15} = \frac{8}{10}$
22. $\frac{11}{7} < \frac{8}{5}$
23. $\frac{6}{10} = \frac{9}{15}$
24. $\frac{8}{13} < \frac{5}{8}$

*25. a.



b. $\frac{3}{2} > \frac{4}{3}$; $\frac{2}{3} < \frac{3}{4}$

$\frac{2}{3} > \frac{5}{9}$; $\frac{3}{2} < \frac{9}{5}$

$\frac{4}{3} < \frac{9}{5}$; $\frac{3}{4} > \frac{5}{9}$

c. $\frac{b}{a} > \frac{d}{c}$

8-4. Mixed Numbers.

No novelties are encountered in this section. Teachers may want to supplement with more problems, especially word problems, depending upon the individual class.

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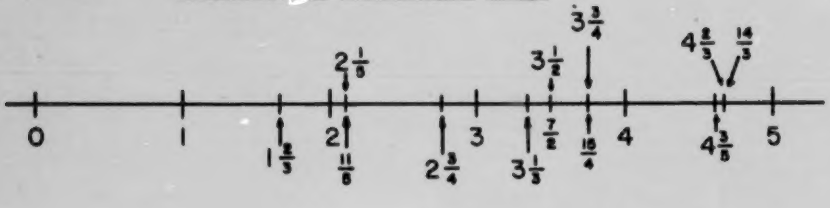
Answers to Exercises 8-4a

1. a. improper
   b. improper
   c. proper
   d. improper
   e. improper
   f. proper
   g. improper
   h. proper
   i. improper
   j. proper
   k. improper
   l. improper
   m. improper
   n. proper
   o. proper

2. e, b proper; l, d, a, c, f improper.

Answers to Exercises 8-4b

1.

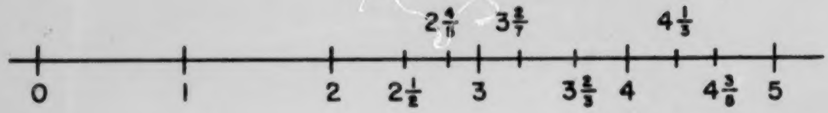

$2\frac{1}{5}$
$3\frac{1}{2}$
$3\frac{3}{4}$
$4\frac{2}{3}$
$\frac{14}{3}$
0
1
$1\frac{2}{3}$
2
$\frac{11}{5}$
$2\frac{3}{4}$
3
$3\frac{1}{3}$
$\frac{7}{2}$
$\frac{15}{4}$
4
$4\frac{3}{5}$
5

2. a. $1 < \frac{3}{2} < 2$
   b. $0 < \frac{3}{4} < 1$
   c. $2 < \frac{8}{3} < 3$
   d. $6 < \frac{70}{11} < 7$
   e. $3 < \frac{24}{7} < 4$
   f. $10 < \frac{81}{8} < 11$
   g. $10 < \frac{111}{11} < 11$
   h. $6 < \frac{33}{5} < 7$
   i. $9 < \frac{84}{9} < 10$
   j. $7 < \frac{57}{8} < 8$

3. a. $1\frac{1}{6}$
   b. $4\frac{2}{3}$
   c. $6\frac{1}{2}$
   d. $2\frac{1}{4}$
   e. $3\frac{6}{7}$
   f. $8\frac{2}{5}$
   g. $21\frac{2}{3}$
   h. $11\frac{3}{4}$
   i. $10\frac{7}{16}$
   j. $31\frac{1}{14}$
   k. $10\frac{19}{21}$
   l. $2\frac{93}{151}$
   m. $1\frac{1}{408}$
   n. $4\frac{34}{111}$
   o. $1\frac{147}{209}$
   p. $7\frac{16}{77}$

Answers to Exercises 8-4c

1.


$2\frac{4}{5}$
$3\frac{2}{7}$
$4\frac{1}{3}$
0
1
2
$2\frac{1}{2}$
3
$3\frac{2}{3}$
4
$4\frac{3}{8}$
5

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2. a. $\frac{7}{4}$
b. $\frac{5}{2}$
c. $\frac{11}{3}$
d. $\frac{13}{5}$
e. $\frac{23}{5}$
f. $\frac{23}{7}$
g. $\frac{13}{3}$
h. $\frac{59}{9}$
i. $\frac{67}{8}$
j. $\frac{75}{7}$
k. $\frac{65}{12}$
l. $\frac{58}{5}$
m. $\frac{100}{7}$
n. $\frac{145}{8}$
o. $\frac{442}{21}$
p. $\frac{362}{19}$

3. a. $3\frac{3}{4}$
b. $5\frac{11}{15}$
c. $4\frac{17}{35}$
d. $2\frac{1}{6}$
e. $2\frac{3}{20}$
f. $7\frac{11}{30}$
g. $1\frac{5}{6}$
h. $\frac{9}{10}$
i. $11\frac{1}{6}$
j. $8\frac{8}{35}$
k. $1\frac{8}{9}$
l. $4\frac{11}{24}$

4. No, there is $\frac{1}{4}$ cup too much.

5. 26 miles.

*6. $2\frac{7}{8}$ gallons.

Answers to Exercises 8-4d

1. a. $8\frac{1}{3}$
b. $3\frac{1}{4}$
c. $21\frac{1}{4}$
d. $2\frac{18}{35}$
e. 55
f. $\frac{3}{7}$
g. $1\frac{2}{3}$
h. $\frac{36}{55}$
i. $6\frac{2}{5}$

2. a. yes
b. $1\frac{3}{4}$ yds. were left over.

3. $3.06

4. 8 hours

5. 21 jars

6. $1\frac{1}{2}$ miles

---

page 294-297: 8-5,8-6

8-5. Complex Fractions.

The definition of a fraction is extended here to include expressions of the form $\frac{a}{b}$ where a and b are rational numbers. The actual form of a and b is not significant. Their numerals may be fractions, mixed numbers, or decimals. Until we proved that the rational numbers were closed under division, it was necessary to limit the idea of the fraction, $\frac{a}{b}$, to a and b whole numbers and $b \neq 0$. Now that we know that division of rational numbers is closed, the symbol, $\frac{a}{b}$, where a and b are themselves rational numbers is meaningful and $\frac{a}{b}$ is a name for a rational number. Much later in algebra the concept of a fraction $\frac{a}{b}$ will be further extended to include all symbols of the form $\frac{a}{b}$, where a and b are real numbers. Then expressions like $\frac{\pi}{2}$, or $\frac{\sqrt{2}}{2}$ will be called fractions also.

Answers to Exercises 8-5

1. a. $\frac{8}{9}$
   b. $\frac{10}{7}$ or $1\frac{3}{7}$
   c. $\frac{5}{3}$ or $1\frac{2}{3}$
   d. $\frac{8}{15}$
   e. $\frac{16}{9}$ or $1\frac{7}{9}$
   f. 4
   g. $\frac{2}{3}$
   h. $\frac{9}{4}$ or $2\frac{1}{4}$
   i. $\frac{35}{36}$

2. a. $\frac{2}{3}$
   b. $\frac{24}{35}$
   c. $\frac{5}{6}$
   d. $\frac{50}{27}$ or $1\frac{23}{27}$
   e. $\frac{4}{5}$
   f. 1

8-6. Equivalent Fractions.

This is the first of several applications of the Comparison Property to the solution of proportions. The word proportion is not introduced in the student text until Chapter 10 on ratios. The solution of proportions is a recurring theme of Volume I and Volume II. It is used:

1. to find equivalent fractions (Chapter 8).

2. to find decimal representation of fractions (Chapter 9).
3. to solve ratio problems (Chapter 10).
4. to solve all types of percentage problems (Chapter 10)
5. to find equivalent measures (Chapter 11).
6. to convert between the metric system and the English system (Chapter 21).

Proportions arise very naturally and frequently in most sciences; chemistry, physics, biology, and engineering, in particular. The early introduction and solution of this type of equation should be extremely valuable and helpful to students.

The form in which the proportion is solved is of great importance. The teacher should insist that at least three steps be written on the pages, one beneath the other. Special vigilance is required to prevent incorrect use of equality signs.

In this section juxtaposition is introduced with the symbol $5n$ to stand for $5 \cdot n$ or $n \cdot 5$. Some students may ask if $n5$ could be used also. Preference for $5n$ is a matter of convention.

We have included problems on expressing fractions with denominator 10 to help the student when he needs to express rational numbers as decimals. The student should see that we can only approximate the location of $\frac{3}{7}$ if we locate it on a number scale divided into tenths of a unit. Exercise Set 8-6 can be easily divided into two parts: Problems 1-7, and Problems 8-12.

Answers to Exercises 8-6

1. a. $4n = 18$ b. $n = \frac{18}{4} = \frac{9}{2}$ or $4\frac{1}{2}$
2. a. $6n = 18$ b. $n = \frac{18}{6} = 3$
3. a. $2n = 56$ b. $n = \frac{56}{2} = 28$

4. a. $3n = 100$ b. $n = \frac{100}{3} = 33\frac{1}{3}$

5. a. $10n = 30$ b. $n = \frac{30}{10} = 3$

6. a. $3n = 100$ b. $n = \frac{100}{3} = 33\frac{1}{3}$

7. a. $10n = 30$ b. $n = \frac{30}{10} = 3$

8. $\frac{1}{8} = \frac{n}{100}$, $8n = 100$, $n = \frac{100}{8} = 12\frac{1}{2}$, $\frac{1}{8} = \frac{12\frac{1}{2}}{100}$

9. $\frac{3\frac{1}{3}}{10}$, $\frac{33\frac{1}{3}}{100}$, $\frac{333\frac{1}{3}}{1000}$

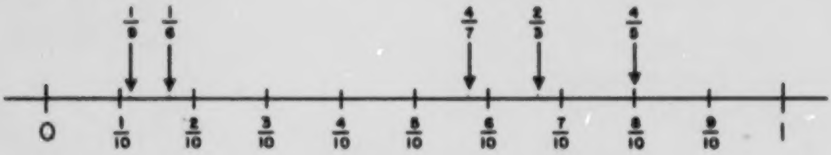
10. a. $\frac{\frac{10}{9}}{10} = \frac{1\frac{1}{9}}{10}$ e. $\frac{8}{10}$

b. $\frac{\frac{20}{3}}{10} = \frac{6\frac{2}{3}}{10}$ f. $\frac{\frac{10}{6}}{10} = \frac{1\frac{2}{3}}{10}$

c. $\frac{\frac{40}{7}}{10} = \frac{5\frac{5}{7}}{10}$ g. $\frac{\frac{70}{3}}{10} = \frac{23\frac{1}{3}}{10} = 2 + \frac{3\frac{1}{3}}{10}$

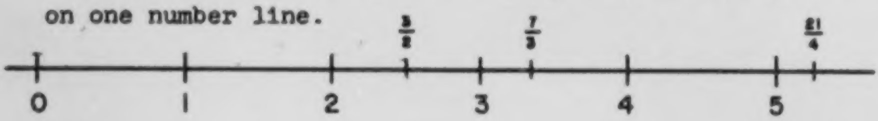
d. $\frac{\frac{210}{4}}{10} = \frac{52\frac{1}{2}}{10} = 5 + \frac{2\frac{1}{2}}{10}$ h. $\frac{15}{10}$

11. We have put all of the points < 1 on this number line:



Note that except for $\frac{4}{5}$ all points are only located approximately.

The points > 1 are located on the following number line. You may wish to have the students put all of the points on one number line.



12. a. $\frac{\frac{100}{9}}{100} = \frac{11\frac{1}{9}}{100} = \frac{11\frac{1}{9}}{10^2}$

b. $\frac{\frac{2000}{3}}{1000} = \frac{666\frac{2}{3}}{1000} = \frac{666\frac{2}{3}}{10^3}$

c. $\frac{\frac{210000}{4}}{10,000} = \frac{52500}{10,000} = \frac{52500}{10^4}$

*8-7. Operations on the Number Line.

So far, our operations with numbers on the number line have been confined to the whole numbers. Here we show that the same geometrical methods of adding and subtracting apply to any rational numbers. So as not to bring in the added difficulty of locating the numbers on the number line, most of the examples in this section have the numbers indicated by letters. We are careful to state that the numbers a, b, etc. are rational numbers, even though the methods apply as well when the numbers are irrational. But the student has not yet encountered the irrational numbers. When points are selected on the number line, it is necessary to specify that these are points representing rational numbers. We will not be ready to treat the question of whether every point on the number line represents a number until we discuss the real numbers in a later course.

Multiplication and division of rational numbers can also be shown geometrically on the number line, but it is more complicated and not as useful.

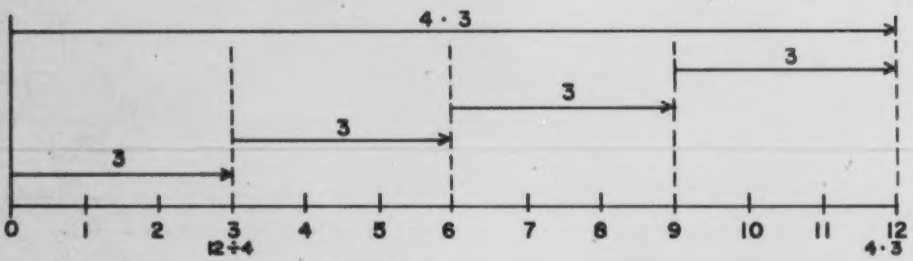
Answers to Exercises 8-7

1. u
2. v
3. s
4. r
5. t
6. t
7. p
8. t
9. q
10. q
11. p
12. t
13. t
14. s
15. v

8-9. Chapter Review.

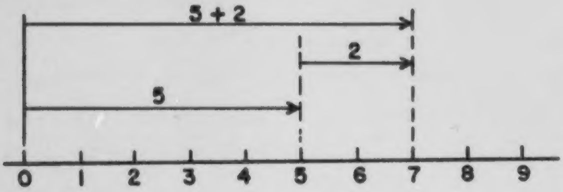
Answers to Exercises 8-9

1. a.


4 · 3
3
3
3
3
0
1
2
3
12 ÷ 4
4
5
6
7
8
9
10
11
12
4 · 3

b. Same diagram for 12 ÷ 4.

c.


5 + 2
2
5
0
1
2
3
4
5
6
7
8
9

d. Same diagram for 7 - 5.

2. a. $2\frac{1}{2} + 2\frac{1}{2} = 5$ or $\frac{5}{2} + \frac{5}{2} = 5$.

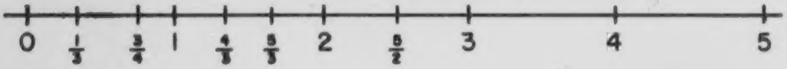
b. $5 - \frac{5}{2} = \frac{5}{2}$.

c. $2 \cdot \frac{5}{2} = 5$

d. $5 \div 2 = \frac{5}{2}$.

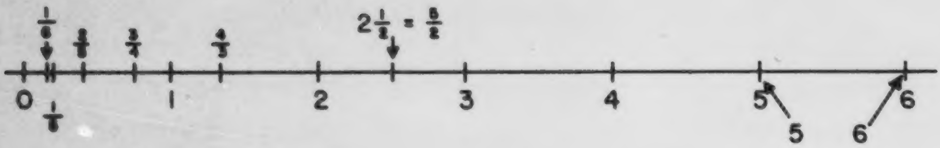
(In general, a diagram will show either addition and its inverse operation or multiplication and its inverse but not both, as in this special case.)

3.


0
1/3
3/4
1
4/3
5/3
2
5/2
3
4
5

4. a. y
b. v
c. x
d. z
e. v
f. w
g. x
h. x
i. y
j. u
k. x
l. z

5.



6. If a rational number is greater than 1, then when it is expressed as a fraction its numerator is greater than the denominator. Therefore, the reciprocal has the denominator greater than the numerator and the reciprocal is less than 1.

If the rational number is between 0 and 1, then when it is expressed as a fraction its numerator is less than its denominator. Its reciprocal has a denominator less than its numerator and represents a rational number greater than 1.

7. a. $b < e$ c. $m > n$ e. $r < s$
b. $e > m$ d. $n < r$ f. $s > u$

8. a. $\frac{2}{3} < \frac{3}{4}$ c. $\frac{3}{7} > \frac{2}{5}$ e. $\frac{4}{11} > \frac{1}{3}$
b. $\frac{3}{6} = \frac{2}{4}$ d. $\frac{6}{9} = \frac{4}{6}$ f. $\frac{16}{7} > \frac{25}{11}$

9. a. $\frac{3}{8} < \frac{2}{5}$ c. $\frac{13}{7} < \frac{15}{8}$ e. $\frac{6}{5} > \frac{7}{6}$
b. $\frac{8}{12} = \frac{10}{15}$ d. $\frac{2}{9} < \frac{3}{10}$ f. $\frac{6}{7} > \frac{5}{6}$

10. $\frac{3}{10}$ 11. 2 12. $\frac{4}{15}$

13. Brainbuster: Each part may be used in the following part.

a. $1 + \cfrac{1}{1 + \frac{1}{2}} = 1 + \cfrac{1}{\frac{3}{2}} = 1 + \frac{2}{3} = \frac{5}{3}$

b. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}} = 1 + \cfrac{1}{\frac{5}{3}} = 1 + \frac{3}{5} = \frac{8}{5}$

c. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}} = 1 + \cfrac{1}{\frac{8}{5}} = 1 + \frac{5}{8} = \frac{13}{8}$

8-10. Cumulative Review.

Answers to Exercises 8-10

1. addition
2. inverse
3. counting, even counting
4. subtracting
5. none or zero
6. a. {6, 12, 18, 24}
   b. This is the empty set
7. {32, 36, 40, 44, 48}
8. 70

9. a. 1 b. $\frac{1}{2}$ c. $\frac{9}{8}$

10. a. $\frac{1}{7}$ c. $\frac{92}{7}$ e. $\frac{21}{8}$
    b. $\frac{1}{19}$ d. $\frac{8931}{6}$ f. 7

11. a. Δ AFC, Δ BDE, Δ BGC
    b. points G and C
    c. the empty set
    d. $\angle$GBC
    e. point A
    f. $\overline{AD}$

12. $\overleftrightarrow{EB}$ and $\overleftrightarrow{CE}$ lie in plane CEB. Since A is on $\overleftrightarrow{EB}$ and D is on $\overleftrightarrow{CE}$, then $\overleftrightarrow{AD}$ is in the plane CEB. If a line contains two different points of a plane, it lies in the plane.

Sample Test Questions for Chapter 8

This set of questions should not be used as a chapter test. Teachers should construct their own tests using carefully selected items from those given here and from items of their own.

True or False:

(T) 1. $\frac{8\frac{1}{3}}{100} = \frac{1}{12}$

(F) 2. If one fraction has a larger numerator than that of a second fraction, the number represented by the first fraction is always larger than the number represented by the second fraction.

(T) 3. The diagrams for the problems $9 \cdot 3$ and $27 \div 3$ are the same.

(F) 4. A whole number can never have a fraction as a name.

(F) 5. If two fractions have the same denominator, the numbers they represent are always equal.

(T) 6. $\frac{\frac{7}{3}}{\frac{1}{2}}$ can be represented as a fraction with numerator and denominator as counting numbers.

(T) 7. A fraction is a numeral indicating the quotient of two numbers, with denominator different from zero.

(T) 8. $\frac{9}{11} = \frac{\frac{900}{11}}{100}$

(F) 9. The reciprocal of $\frac{13}{19}$ is $\frac{1}{\frac{19}{13}}$

(T) 10. $\frac{19}{21} < \frac{21}{22}$

Multiple Choice:

b. 1. If $4 \cdot 5 > 2 \cdot 2$ then

a. $\frac{4}{2} < \frac{2}{5}$

b. $\frac{4}{2} > \frac{2}{5}$

c. $\frac{4}{2} = \frac{2}{5}$

d. none of these

c. 2. If $51 \cdot 3 = 17 \cdot 9$ then

a. $\frac{51}{9} > \frac{3}{17}$

b. $\frac{51}{9} < \frac{3}{17}$

c. $\frac{51}{9} = \frac{17}{3}$

d. none of these

c. 3. We can change the denominator of the fraction $\frac{\frac{2}{3}}{\frac{4}{5}}$ to the number "1" without changing the value of the fraction by

a. adding $\frac{5}{4}$ to the numerator and denominator
b. subtracting $\frac{5}{4}$ from the numerator and denominator
c. multiplying both numerator and denominator by $\frac{5}{4}$
d. dividing both numerator and denominator by $\frac{5}{4}$
e. none of these

Completion:

1. In each case below insert one of the symbols, <, =, >, so as to make the statement true:

		Ans.			Ans.
a.	$\frac{7}{8}$ $\frac{5}{8}$	>	e.	$\frac{6}{20}$ $\frac{11}{35}$	<
b.	$\frac{3}{8}$ $\frac{3}{9}$	>	f.	$\frac{0}{5}$ $\frac{0}{3}$	=
c.	$\frac{3}{2}$ $\frac{2}{3}$	>	g.	$\frac{19}{20}$ $\frac{18}{19}$	>
d.	$\frac{9}{9}$ $\frac{7}{7}$	=			

2. A crafts class needs a type of decoration that sells for $7\frac{1}{2}$ cents a foot in one shop and at 3 feet for 25 cents in another shop. How much can be saved on each foot at the cheaper price?

Ans. At the second shop the price is $8\frac{1}{3}$ cents for a foot, hence $\frac{5}{6}$ of a cent per foot can be saved by buying at the first shop.

3. A group of seventh graders have promised to collect 50 pounds of scrap metal. They have $36\frac{7}{8}$ pounds; how much more must they collect to keep their promise?

Ans. $13\frac{1}{8}$ pounds

4. Tom needs four pieces of wood $2\frac{3}{4}$ feet long for the legs of a table. Boards from which this wood can be cut come in the following lengths: 8 feet, 10 feet, 12 feet. What length board should he get? How much will be left over?

   Ans. He needs 11 feet; hence he should get the 12 foot board. He will have one foot left over.

5. There are 40 questions on a test. If all questions are given the same value and if a perfect paper gets a grade of 100, how much should each question count? How many questions would a student have to answer correctly to get a grade of 90 or better?

   Ans. $2\frac{1}{2}$, 36

6. When a merchant buys candy bars, he pays 40 cents for boxes holding 25 bars. If he sells them at 2 bars for 5 cents, what is his profit on each bar?

   Ans. Each bar costs 1.6 cents, he sells them for $2\frac{1}{2}$ cents each. Hence his profit is 0.9 cents each. This can also be found using fractions exclusively.

Chapter 9

DECIMALS

The major aims of this chapter are:

1. To explain decimal notation.
2. To develop and rationalize the rules for performing arithmetic operations with decimals.
3. To develop a method for writing rational numbers as decimals.
4. To show that every rational number may be expressed as a repeating decimal.
5. To develop the concept of rounding numbers.

The exposition in this chapter does not presuppose any prior knowledge of decimals on the part of the student. Where this is not the case, the teacher may find it convenient to cover the first part of the chapter quite rapidly. However, if this is the students' first exposition to decimals, time should be taken to cover the initial portions of the chapter quite carefully as well as to discuss the rationale of the fundamental operation with decimals.

For the very slow student, Section 9-3 might be omitted without great harm. However, experience indicates that even the slow learner seems to enjoy the topic of repeating decimals.

9-1. Decimal Notation.

This section opens with a discussion of decimal notation and expanded form. Since the material on expanded form was first introduced in Chapter 2, it may be well to devote some time to a review of this topic before introducing decimals.

The decimal point is introduced as a punctuation mark which separates the whole number places from the decimal places. As an outcome of this lesson we expect the student to recognize that the place values in a number, as you read from left to right, are decreasing powers of 10.

We also want the student to understand the meaning of place value. For example, he should see that the digit 3 represents three-tenths in each of the following: 0.3 ; 7.3 ; 8.35 ; 29.376 ; etc.

There is another way to introduce decimal notation which is not included in the students' text but which you may wish to try:

Consider the numeral 4659 in expanded form:

(a) $4(10^3) + 6(10^2) + 5(10) + 9(1)$

Multiply this number by $\frac{1}{10}$:

(b) $\frac{1}{10}[4(10^3) + 6(10^2) + 5(10) + 9] =$
$4(10^2) + 6(10) + 5(1) + 9(\frac{1}{10})$

Now note that multiplying the number in (a) by $\frac{1}{10}$ is the same as dividing it by 10. Then each digit in (b) represents a number which is $\frac{1}{10}$ of the number represented by the corresponding digit in (a). For example, the digit 6 in (b) represents 6 × 10 whereas the digit 6 in (a) represents 6 × 100.

In a similar fashion, we can multiply (a) by $\frac{1}{100}$:

(a) $4(10^3) + 6(10^2) + 5(10) + 9(1)$

(c) $4(10) + 6(1) + 5(\frac{1}{10}) + 9(\frac{1}{10^2})$

Now each digit in (c) represents a number which is $\frac{1}{100}$ of the number represented by the corresponding digit in (a).

Next we consider how to write the numbers (b) and (c) in positional notation. Clearly 4659 is incorrect. We decide to "invent" a decimal point and write (b) as 465.9 and (c) as 46.59.

Answers to Exercises 9-1a
(Class Discussion)

1. a. 0.3
   b. 0.27
   c. 0.389
   d. 1.9
   e. 25.7
   f. 81.35

2. a. four tenths
   b. thirty-two and seven tenths
   c. twenty-eight hundredths

d. three and forty-seven hundredths

e. two hundred fifty-seven thousandths

f. seventeen and nine hundred thirty-five thousandths

3. a. $\frac{55}{100}$

b. $\frac{21}{100}$

c. $\frac{4}{10}$

d. $\frac{13}{1000}$

e. $\frac{105}{1000}$

f. $\frac{12}{1000}$

g. $\frac{3}{100}$

h. $\frac{84}{100}$

i. $\frac{64}{1000}$

j. $\frac{25}{100}$

4. a. $\frac{75}{100}$

b. $\frac{5}{10}$

c. $\frac{30}{100}$ or $\frac{3}{10}$

d. $\frac{18}{1000}$

e. $\frac{36}{100}$

5. a. $5 + \frac{2}{10} + \frac{3}{10^2}$

b. $3 + \frac{7}{10} + \frac{9}{10^2}$

c. $28 + \frac{0}{10} + \frac{5}{10^2}$

d. $75 + \frac{9}{10} + \frac{1}{10^2}$

6. a. .38

b. .79

c. 5.01

d. 23.95

Answers to Exercises 9-1b

1. a. 4,967

b. 5,618.3

c. 245.61

d. 804.359

e. 5.24

f. 0.483

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g. 372.06

h. 0.0206

i. 3.507

2. a. $7(10^3) + 8(10^2) + 6(10) + 2$

b. $4(10^2) + 3(10) + 7 + 9(\frac{1}{10})$

c. $2(10) + 8 + 6(\frac{1}{10}) + 4(\frac{1}{10^2})$

d. $3(10^2) + 4(10) + 7 + 1(\frac{1}{10}) + 5(\frac{1}{10^2})$

e. $9(10) + 6 + 3(\frac{1}{10}) + 7(\frac{1}{10^2}) + 2(\frac{1}{10^3})$

f. $2 + 4(\frac{1}{10}) + 6(\frac{1}{10^2}) + 5(\frac{1}{10^3})$

g. $3(\frac{1}{10}) + 8(\frac{1}{10^2}) + 4(\frac{1}{10^3})$

h. $1(\frac{1}{10^2}) + 3(\frac{1}{10^3})$

i. $2(10) + 4 + 0(\frac{1}{10}) + 9(\frac{1}{10^2})$

j. $3(\frac{1}{10^3}) + 9(\frac{1}{10^4})$

3. a. six hundred fifty-eight
b. three and two tenths
c. four and seventy-three hundredths
d. fifty-eight and twenty-nine hundredths
e. seven hundred fifty-nine and six tenths
f. forty-eight and seven hundredths
g. three and two hundred nine thousandths
h. thirty-seven and one hundred six ten-thousandths

4. a. 5.52

b. 762.9

c. 300.52

d. .14

e. 2.007

f. 60.07

5. a. $7 \cdot 10$

b. $2 \cdot 10^2$

c. $2 \cdot 1$

d. $5 \cdot 10^3$

e. $0 \cdot 10$

f. $5 \cdot 10^2$

*6. .5

*7. $.5 = .6_{\text{twelve}}$

*8. $\frac{5}{6} = .t_{\text{twelve}}$

*9. $$10.011_{\text{two}} = 1(2) + 1(\frac{1}{2^2}) + 1(\frac{1}{2^3}) = 2 + \frac{1}{4} + \frac{1}{8} = 2 + \frac{2}{8} + \frac{1}{8}$$
$$= 2\frac{3}{8} = 2.375$$

9-2. Arithmetic Operations With Decimals.

The purpose of this section is to present the reasons for the manner in which we add, subtract, multiply, and divide decimals.

For addition and subtraction we express the decimals as fractions and make use of the distributive property. This lets us add whole numbers and explains the algorithm that the student uses. Thus in finding the sum $0.73 + 0.84$ we proceed as follows:

$$0.73 + 0.84 = (73 \times \frac{1}{100}) + (84 \times \frac{1}{100})$$
$$= (73 + 84) \times \frac{1}{100}$$ by the distributive property
$$= 157 \times \frac{1}{100}$$

Now this step shows that we add the whole numbers 73 and 84 ; that is, we can ignore the decimal point for a moment. Then we multiply by $\frac{1}{100}$ to get the sum 1.57. This explains the algorithm:

0.73 $\underline{0.84}$ 1.57	Add 73 and 84 ; place the decimal point in the sum beneath those in the addends.

We attempt a "discovery" approach for multiplication by asking the student a set of questions which leads to a method of locating the decimal point. Again we emphasize operations on whole numbers, based on previously established properties.

Division of decimals has always been a difficult topic for students, especially the slow learner. We base our introduction upon the Comparison Property presented in Chapter 8. We then show that division can be done more readily by using the Multiplication Property of 1. At this point the usual algorithm is exhibited as a convenient way of dividing.

Nowhere do we "shift" decimal points. Instead we always make our divisor a whole number through use of the Multiplication Property of 1.

Some students may have been taught the algorithms for computing with decimals in earlier grades. If so, it is still important to present the rationale for these operations.

Once the rationale is understood, time will have to be spent to provide drill on each of the four operations. The teacher may wish to provide more practice material than appears in the text.

Answers to Exercises 9-2a

1. a. .8
   b. 1.32
   c. 1.45
   d. 1.101
   e. 1.00332
   f. 23.30
   g. 49.22

2. a. 0.6
   b. 0.08
   c. 0.26
   d. 0.39
   e. 0.075
   f. 0.375
   g. 1.0045

3. .45 lb.

4. 5.2 seconds

5. 185.8 miles

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6. 14.27 inches

7. 1.75 inches

*8. 3.5

Answers to Exercises 9-2b

1. a. .56
   b. .054
   c. .0015
   d. .0016
   e. .00014
   f. 2.4
   g. .072
   h. .00063
   i. 5.4
   j. .0048

2. a. 2.6
   b. .75
   c. .036
   d. 2.25
   e. 1.421
   f. .1334
   g. .26751
   h. .3840
   i. 205.36
   j. 6.223

3. about 33.4 lbs.

4. 19 oz.

5. 15.6 inches

6. $1,002.00

7. 247.5 ft.

8. about 2.7 mi.

Answers to Exercises 9-2c

1. a. $\frac{1}{7} = \frac{14\frac{2}{7}}{100}$ b. $\frac{2}{3} = \frac{66\frac{2}{3}}{100}$

c. $\frac{1}{9} = \frac{11\frac{1}{9}}{100}$

d. $\frac{4}{3} = \frac{133\frac{1}{3}}{100}$

e. $\frac{10}{9} = \frac{111\frac{1}{9}}{100}$

2. a. $\frac{142\frac{6}{7}}{1000}$

b. $\frac{666\frac{2}{3}}{1000}$

c. $\frac{111\frac{1}{9}}{1000}$

d. $\frac{1333\frac{1}{3}}{1000}$

e. $\frac{1111\frac{1}{9}}{1000}$

3. a. .14

b. .67

c. .11

d. 1.33

e. 1.11

4. a. .143

b. .667

c. .111

d. 1.333

e. 1.111

5. a. 0.03

b. 0.04

c. 0.03

d. 13.3

e. 255

f. 205

g. 2.1

h. 1.8

i. 2.4

j. 2.5

6. (These answers have been rounded to the hundredths place.)

a. 3.60

b. 169.30

c. 26.68

d. 14.11

e. 137.59

f. 11.18

g. 69.79

h. 1.92

i. .94

j. .92

7. (These answers have been rounded to the thousandths place.)

   a. 5.118
   b. .113
   c. 5.417
   d. 61.519
   e. 53.080
   f. .020
   g. 1.453
   h. 1.068
   i. 4.710
   j. 20.313

8. 1.09 lbs.

9. 19.5 mi./gal.

10. $4.52

11. 30 rods

12. $3.45

9-3. Repeating Decimals.

By this time the student already knows how to express a rational number as a decimal to any desired number of places. He can do this by division. The first part of this section reviews this idea, and also points out that it is not necessary to divide when given such fractions as $\frac{7}{10}$, $\frac{23}{100}$, $\frac{3}{20}$, etc.

In some cases it is easier to divide, such as in the cases of fractions like $\frac{1}{3}$, $\frac{4}{11}$, $\frac{23}{37}$, etc.

We then introduce notation and show that every rational number can be written as a repeating decimal. In a starred section of Chapter 20 we show that the converse of this statement is true, namely that every repeating decimal names a rational number. (Furthermore we show there that those decimals which are not repeating are names for irrational numbers.) The very slow student can omit this section without loss of continuity.

Answers to Exercises 9-3a
(Class Discussion)

1. (Answered in student text.)

2. Repetition begins at the seventh decimal place. (.1428571)

3. Many possible answers.

4. Consider the first problem, 16 ÷ 33. The remainder 16 is the same as the original dividend. This remainder has not yet been divided by 33. To do this division involves a repetition of what has already been done. If the process is continued through an additional two partial divisions, the remainder is again 16. If continued through another two partial divisions, the remainder 16 again reappears. The repetition continues endlessly.

   A similar explanation applies for the second problem, 92 ÷ 111. In other words, the remainders begin repeating.

Answers to Exercises 9-3b

1. a. $0.1\overline{1}$
   b. $0.51\overline{1}$
   c. $0.46\overline{6}$
   d. $0.675\overline{675}$
   e. $0.83\overline{3}$
   f. $0.288\overline{288}$

2. a. 0.091
   b. 0.182
   c. .273
   d. .818
   e. 1.273
   f. 2.091

3. a. The decimal numeral for $\frac{2}{11}$ is twice the decimal numeral for $\frac{1}{11}$.
   b. 3 times, 9 times, 14 times, 23 times the decimal numeral for $\frac{1}{11}$.

4. Yes. $0.45\overline{45}$

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5. Yes. $0.63\overline{53}$

6. Yes.

Answers to Exercises 9-3c
(Class Discussion)

1. Yes, 142857

2. Yes.

3. $0.09\overline{09}$

4. No.

5. Yes. By a bar over the block of digits that repeats, that is, $0.09\overline{09}$.

6. Yes.

7. a. Yes.
   b. Yes.
   c. Yes.

8. $0.027\overline{027}$

The following explanation is necessary to complete the Class Discussion Exercises 9-3c. It was not included in the student text for fear of causing the reader any possible further confusion.

In finding the decimal representation for $\frac{1}{7}$, we divide 1 by 7. Since the divisor is 7, each remainder must be one of the set of digits 0, 1, 2, 3, 4, 5, 6. If the remainder is 0, the decimal terminates. If it is not 0, then the remainder must repeat after at most six steps since there are only this many possible remainders left (1, 2, 3, 4, 5, 6). In the case of $\frac{1}{7}$ the remainders, in order, are 3, 2, 6, 4, 5, 1. The seventh remainder must then be either one of these or be 0. Since the seventh remainder is

3 , the decimal representation begins to repeat at this point. From this point on, as shown in the array in the students' text, the entire sequence repeats and does not depend on the preceding computations.

Answers to Exercises 9-3d

1. $\frac{1}{13} = 0.076923\overline{076923}$
   a. Seventh place.
   b. No.
   c. By a bar over the block of digits that repeats.

2. a. $0.3\overline{3}$
   b. $0.16\overline{6}$
   c. $0.1\overline{1}$
   d. $0.36\overline{36}$
   e. $0.46\overline{6}$

3. a. $0.50\overline{0}$
   b. $0.20\overline{0}$
   c. $0.750\overline{0}$
   d. $0.60\overline{0}$
   e. $0.8750\overline{0}$
   f. $0.80\overline{0}$
   g. $0.6\overline{6}$
   h. $0.81\overline{81}$
   i. $0.5\overline{5}$
   j. $0.73\overline{3}$

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*4.

	A	B	C	D	E	F
a. $\frac{3}{5}$	$\frac{6}{10}$	[✓ .6]	$\frac{60}{100}$	[✓ .60]	$\frac{600}{1000}$	[✓ .600]
b. $\frac{5}{9}$	$\frac{5\frac{5}{9}}{10}$	.6	$\frac{55\frac{5}{9}}{100}$	.56	$\frac{555\frac{5}{9}}{1000}$	.556
c. $\frac{3}{2}$	$\frac{15}{10}$	[✓ 1.5]	$\frac{150}{100}$	[✓ 1.50]	$\frac{1500}{1000}$	[✓ 1.500]
d. $\frac{36}{25}$	$\frac{14\frac{2}{5}}{10}$	1.4	$\frac{144}{100}$	[✓ 1.44]	$\frac{1440}{1000}$	[✓ 1.440]
e. $\frac{1}{8}$	$\frac{1\frac{1}{4}}{10}$	.1	$\frac{12\frac{1}{2}}{100}$	.13	$\frac{125}{1000}$	[✓ .125]
f. $\frac{1}{6}$	$\frac{1\frac{2}{3}}{10}$	.2	$\frac{16\frac{2}{3}}{100}$	.17	$\frac{166\frac{2}{3}}{1000}$	.167
g. $\frac{3}{4}$	$\frac{7\frac{1}{2}}{10}$	.8	$\frac{75}{100}$	[✓ .75]	$\frac{750}{1000}$	[✓ .750]

9-4. Rounding Decimal Numbers.

We begin this section by showing the reasonableness of rounding large numbers and relating the process of rounding to the number line. This procedure is then applied to decimals. We also try to point out that the agreement to round to the larger of two numbers when we are given a number halfway between is an arbitrary one.

Answers to Questions in Examples in Section 9-4

Example 1. The point halfway between 20 and 30 is 25. We round 23 to 20, because it is on the side of 25 which is closer to 20.

Example 2. The point halfway between 600 and 700 is 650. We round 677 to 700 because it is on the side of 650 which is closer to 700.

Example 3. Answered in student text.

Example 4. Answered in student text.

Example 5. The point halfway between .1800 and .1900 is .1850. We round .1839 to .1800, because it is on the side of .1850 which is closer to .1800.

Answers to Exercises 9-4a

1. a. 300
   b. 300
   c. 1400
   d. 900
   e. 700
   f. 3600
   g. 400
   h. 1000
   i. 5300

2. a. 6,000
   b. 3,000
   c. 3,000
   d. 15,000
   e. 63,000
   f. 144,000
   g. 145,000
   h. 145,000
   i. 326,000

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3. a. 730,000
   b. 160,000
   c. 70,000
   d. 200,000
   e. 570,000
   f. 90,000

Answers to Exercises 9-4b

1. a. the thousandths digit, 7 , is greater than 5
   b. the thousandths digit, 1 , is less than 5
   c. the thousandths digit, 4 , is less than 5
   d. the thousandths digit, 4 , is less than 5

2. a. 48.36
   b. 0.52
   c. 35.02
   d. 6.01
   e. 0.01
   f. 0.10

3. a. 16.4
   b. 48.7
   c. 108.1
   d. 0.1
   e. 0.1
   f. 1.2
   g. 3.1
   h. 68.1
   i. 43.0

4. a. 4.049
   b. 17.107
   c. 0.001
   d. 0.002
   e. 185.731
   f. 62.912

5. a. 0.6
   b. 0.3
   c. 0.4
   d. 0.8
   e. 2.1
   f. 3.3

6. a. 0.53
   b. 0.67
   c. 0.56
   d. 1.58
   e. 0.71
   f. 0.17

9-6. Chapter Review.

Answers to Exercises 9-6

1. a. $7(10) + 9 + 3(\frac{1}{10})$

   b. $4(10^2) + 5(10) + 3 + 0(\frac{1}{10}) + 8(\frac{1}{10^2})$

   c. $6(10^3) + 0(10^2) + 2(10) + 8 + 3(\frac{1}{10}) + 5(\frac{1}{10^2}) + 7(\frac{1}{10^3})$

   d. $5(10^3) + 7(10^2) + 3(10) + 9 + 2(\frac{1}{10}) + 0(\frac{1}{10^2}) + 5(\frac{1}{10^3})$

2. a. 352.7

   b. 4,937.38

   c. 8,094.109

3. a. Fifty-four and three tenths
   b. One hundred sixty-nine and five hundredths

4. a. 6.9

   b. 90.06

5. a. 0.30

   b. 0.83

   c. 1.38

   d. 0.22

   e. 2.75

   f. 0.67

6. a. 0.47 , 0.44

   b. 0.36 , 0.35

   c. 0.32 , 0.28

7. a. 3.55

   b. 4.661

8. a. 0.45

   b. 1.129

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9. a. 5.18
   b. 4.456

10. a. 0.0315
    b. 0.0056

11. a. 0.1897
    b. 179.8

12. a. $0.27\overline{27}$
    b. $0.7\overline{7}$
    c. $0.\overline{384615}$

13. a. 400
    b. 1,300
    c. 900
    d. 12,500

14. a. 7,000
    b. 485,000
    c. 77,000
    d. 5,000

15. a. 2.5
    b. 385.1
    c. 96.6
    d. 1,043.1

16. a. 3.27
    b. 489.06
    c. 0.02
    d. 5,829.14

17. a. 25.095
    b. 3.126
    c. 47.695
    d. 78.044

18. a. 0.88
    b. 0.24

9-7. Cumulative Review.

Answers to Exercises 9-7

1. Distributive property.

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2. All are correct.

3. $7 \times 7 \times 7 \times 7$; 2401

4. a. True
   b. False
   c. True
   d. False
   e. True

5. a. $\angle RST$
   b. the vertex
   c. the interior of $\angle RST$

6. a. $\overleftrightarrow{AD}$
   b. There are several; for example, $\overleftrightarrow{AD}$ and $\overleftrightarrow{EG}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{FH}$, etc.
   c. the empty set
   d. $\angle EFH$

7. a. 4
   b. $3\frac{3}{5}$ or $\frac{18}{5}$
   c. $3\frac{1}{2}$ or $\frac{7}{2}$
   d. $8\frac{7}{10}$
   e. $1\frac{13}{16}$
   f. 2
   g. 2
   h. $1\frac{5}{6}$
   i. $8\frac{7}{8}$

8. a. 2.125
   b. 5.25
   c. 0.9
   d. 0.875
   e. 1.75

9. $53.89

10. 3031 Check: $3031 \times 28 = 84,868$

11. a. $\frac{1}{2}$
    b. 2

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c. 0

d. $\frac{4}{3}$ or $1\frac{1}{3}$

e. $\frac{5}{7}$

12. a. $\frac{8}{12} = \frac{2}{3}$

b. $\frac{3}{4} > \frac{2}{3}$

c. $\frac{8}{12} = \frac{16}{24}$

d. $\frac{7}{8} < \frac{8}{9}$

e. $\frac{6}{7} > \frac{11}{13}$

Sample Test Questions for Chapter 9

Note: Teachers should construct their own tests, using carefully selected items from those given here and from items of their own. Careful attention should be given to difficulty of items and time required to complete the tests.

1. True - False

(F) a. 502 written in expanded form is $5(10^2) + 1(10) + 2$.

(F) b. $\frac{1}{3} = 0.13333...$

(T) c. $0.3 \times 0.03 = 0.009$

(T) d. $.2 \div .08 = 2.5$

(F) e. $3\frac{1}{8}$, rounded to the nearest tenth, is equal to 3.2.

2. Complete the following:

a. In the decimal numeral 9384.562 the digit 9 occupies the __________ place. (thousands)

b. Written in positional notation the numeral $7(10^3) + 5(10) + 3(\frac{1}{10^2})$ is __________. (7,050.03)

c. The difference between the sum of 1.05 and 0.75 and the sum of 0.5 and 0.125 is _______. (1.175)

d. The decimal numeral for the rational number $\frac{1}{11}$ is _______. $(0.09\overline{09})$

e. The decimal numeral 437.454 rounded to the nearest tenth is ________. (437.5)

3. Write each of the following in words:

a. 659.03 (Six hundred fifty-nine and three hundredths)	b. 1,248.409 (One thousand two hundred forty-eight and four hundred nine thousandths)

4. Write in decimal form:

a. Seventy-six and three hundredths (76.03)

b. Five hundred two and forty-eight thousandths (502.048)

5. Perform the operations indicated:

a. 3.05 + 2.48 (5.53)

b. 6.015 - 3.999 (2.016)

6. When the Clark family started on a trip, the speedometer on their car registered 15,467.8 miles. At the end of the first day of driving it registered 15,802.1 miles. At the end of the second day of driving it registered 16,189.4 miles. How many miles did the Clark family travel on the first day? on the second day? (334.3 miles, 387.3 miles)

7. Mr. Brown knows that, on the average, he drives 18.7 miles on each gallon of gas. He has 7.5 gallons left in his tank. How far can he drive on this quantity of gasoline? (140.25 miles)

8. An airliner used 1 gallon of gasoline for each 1.8 miles of flight. How many gallons of gasoline will the airliner use in a flight of 837 miles? (465 gallons)

9. Perform the indicated operations:

a. $.0658 \times 375$ (24.675)

b. $21 \div .024$ (875)

10. Find a decimal numeral for each of the following:

a. $\frac{5}{2}$ (2.5) b. $\frac{9}{8}$ (1.125) c. $\frac{17}{20}$ (0.85)

11. Find a decimal numeral for each of the following. Round the result to two decimal places.

a. $\frac{9}{13}$ (0.69) b. $\frac{3}{17}$ (0.18) c. $\frac{9}{7}$ (1.29)

12. Write decimal numerals for each of the following. Use a horizontal bar to indicate the block of digits that repeats.

a. $\frac{11}{12}$ ($0.91\overline{6}$)

b. $\frac{5}{24}$ ($0.208\overline{3}$)

c. $\frac{10}{11}$ ($0.90\overline{90}$)

d. $\frac{6}{7}$ ($0.\overline{857142}$)

13. Which of the following represents the largest number?

(A) .276

(B) .076

(C) .006

(D) .206

(E) .267

(A)

14. Which of the following expresses $\frac{3}{11}$ as a repeating decimal?

(A) .27

(B) $.27\overline{27}$

(C) .027

(D) .2727

(E) $.0\overline{27}$

(B)

15. Consider the following exercise:

$$0.73 + 0.84 = (73 \times \frac{1}{100}) + (84 \times \frac{1}{100})$$

$$= (73 + 84) \times \frac{1}{100}$$

$$= 157 \times \frac{1}{100}$$

$$= 1.57$$

What property has been used to add these two numbers?

(A) Commutative property for addition

(B) Commutative property for multiplication

(C) Associative property for addition

(D) Associative property for multiplication

(E) Distributive property ____(E)____

Chapter 10

RATIO AND PERCENT

The purpose of Sections 1 and 2 is to give a meaningful introduction to ratio and proportion. The most important concept of this chapter is proportion. A proportion arises when two quantities are compared by measurement in different situations. When a physical or mathematical law can be written as a proportion, then this law can be used to deduce new information from old. Thus, in the example of Section 10-2, shadow length is proportional to height. When we know the constant ratio and the shadow length, the height can be computed.

The use of the notation $a:b$ should not be encouraged but the teacher may point out that the notation $a:b$ means $a \div b$; and $\frac{a}{b}$ means $a \div b$.

The text explains carefully that a ratio is a comparison of two *numbers*. However, we abbreviate frequently. For example:

"The ratio of the length to the width of a rectangle" really means "the ratio of the *number* of units in the length to the *number* of units in the width."

"The ratio of John's height to Mary's height" really means "the ratio of the *number* of inches (or feet) in John's height to the *number* of inches (or feet) in Mary's height."

Similarly, we may speak of the ratio of 2 inches to 5 inches, $\frac{2}{5}$, whereas we really mean the ratio of the *number* of inches in 2 inches to the *number* of inches in 5 inches.

The question of units is not developed in detail in this text, and probably should not be done at this grade level. Normally, the context of the problem will indicate the manner in which units are to be compared.

Let us consider the illustration given in the text concerning the map drawn so that "1 inch = 200 miles". Actually this is an incorrect use of the "=" sign and is a sort of mathematical slang. As pointed out in the student text, it really means that one inch on the map *represents* 200 miles on the ground. Here we find the ratio $\frac{1}{200}$ is more convenient than $\frac{1}{12,672,000}$. But when we use the ratio $\frac{1}{200}$ we must be careful to state that we are talking about the number of inches on the map to the number of miles on the ground.

Let's set up a proportion to find the distance between two towns which are 3 inches apart on the map. Let x = the number of miles apart these towns actually are on the ground. Then:

$$\frac{1}{200} = \frac{3}{x} \qquad \left(\frac{1 \text{ inch}}{200 \text{ miles}} = \frac{3 \text{ inches}}{x \text{ miles}}\right)$$

Note that we have set up two equal ratios, each one comparing inches with miles. Our answer, $x = 600$, will be in terms of number of miles. We could have obtained the same result by means of the following proportion:

$$\frac{1}{3} = \frac{200}{x} \qquad \left(\frac{1 \text{ inch}}{3 \text{ inches}} = \frac{200 \text{ miles}}{x \text{ miles}}\right)$$

One needs to be cautious, however, in setting up the proportion. Note that the numerators tell us that "1 inch = 200 miles", whereas the denominators state "3 inches = x miles".

Now if we use the ratio $\frac{1}{12,672,000}$ we may let d = the number of inches apart the two towns actually are on the ground:

$$\frac{1}{12,672,000} = \frac{3}{x} \qquad \left(\frac{1 \text{ inch}}{12,672,000 \text{ inches}} = \frac{3 \text{ inches}}{x \text{ inches}}\right)$$

Our answer, $x = 38,016,000$, will be in terms of inches. The two answers (600 miles and 38,016,000 inches) are equivalent. The units we use in our proportion depend upon the answer we wish to obtain. Thus, in Chapter 21, a proportion is used to convert 5 inches to centimeters as follows:

$$\frac{1}{2.54} \approx \frac{5}{x} \quad \left(\frac{1 \text{ inch}}{2.54 \text{ centimeters}} \approx \frac{5 \text{ inches}}{x \text{ centimeters}}\right)$$

(1 inch is approximately equal to, $\approx$, 2.54 centimeters)

We could also have used the proportion:

$$\frac{1}{5} \approx \frac{2.54}{x} \quad \left(\frac{1 \text{ inch}}{5 \text{ inches}} \approx \frac{2.54 \text{ centimeters}}{x \text{ centimeters}}\right)$$

When we write a ratio we compare two numbers. In setting up a proportion we need to be cautious about the manner in which units are compared.

Answers to Exercises 10-1a

(Class Discussion)

1. a. $\frac{1}{3}$ b. $\frac{1}{4}$ c. 4

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2. a. $\frac{5}{8}$ c. $\frac{3}{8}$
   b. $\frac{5}{8}$ d. $\frac{3}{5}$

3. a. 75 miles b. $2\frac{1}{2}$ inches

Answers to Exercises 10-1b

1. $\frac{48}{64} = \frac{3}{4}$

2. $\frac{90}{30} = \frac{3}{1} = 3$

3. $\frac{3}{5}$

4. $\frac{5}{3}$

5. $\frac{30}{10} = \frac{3}{1} = 3$

6. 20¢ , $\frac{20}{10} = \frac{2}{1} = 2$

7. $\frac{2}{8} = \frac{1}{4}$

8. $\frac{6}{8} = \frac{3}{4}$

9. $\frac{10}{80} = \frac{1}{8}$

10. a. $\frac{9}{7}$ c. $\frac{3}{10}$
    b. $\frac{5}{8}$ d. $\frac{2}{1} = 2$

11. a. $\frac{1}{20}$
    b. 85 miles

12. 24" long, 16" wide

13. a. $\frac{1}{300}$ b. 3600 miles

c. $\frac{5}{2}$ in. or $2\frac{1}{2}$ in.

14. a. 520 miles per hour

b. $\frac{2600}{5}$ or 520

15. 40 miles per hour

Answers to Exercises 10-1c

1. Depends on height of individual. Length of shadow is $\frac{2}{3} \times$ height.

2.

Object	Shadow Length	Height	Ratio
Garage	3 ft.	8 ft.	$\frac{3}{8}$
Clothes pole	36 in.	96 in.	$\frac{3}{8}$
Tree	$7\frac{1}{2}$ ft.	20 ft.	$\frac{3}{8}$
Flagpole	54 in.	144 in.	$\frac{3}{8}$
Fence	$11\frac{1}{4}$ in.	30 in.	$\frac{3}{8}$

3. a. $\frac{2}{9}$

b. $\frac{48}{40} = \frac{6}{5}$

c. $\frac{175}{125} = \frac{7}{5}$

d. $\frac{65}{100} = \frac{13}{20}$

4. $\frac{c}{d}$

5. $\frac{s}{66}$

6. $\frac{2k}{7m}$

7. $\frac{40}{w}$

10-2. Proportion.

Proportion is a new concept. Essentially it has its origin in physical examples. If four quantities are proportional, then any two ratios of corresponding values of these quantities are equal, since each ratio is equal to the "constant of proportionality." The expression, "constant of proportionality," is not used in the student text. We simply say that the ratios are equal.

The Comparison Property from Chapter 8 is necessary in the solutions of the problems. It may be wise to review its meaning and application.

Good mathematics involves good organization of written work. Encourage the students to follow the form demonstrated in the examples with the successive steps written in vertical arrangement and the equal signs in a straight column. It is not necessary to write the reasons in every case. Knowing them is necessary. Since percent problems may often be written as proportions, care in solving proportions in this section will make the solution of percent problems easier in the later sections.

Answers to Exercises 10-2

1. a. $\frac{3}{10}$
   b. $\frac{36}{96} = \frac{3}{8}$
   c. $\frac{11}{30}$

2. a. 9 lbs.
   b. $\frac{15}{24}$ or $\frac{5}{8}$
   c. $\frac{9}{24}$ or $\frac{3}{8}$
   d. $\frac{15}{9}$ or $\frac{5}{3}$

3. 15 lbs. since $\frac{5}{3} = \frac{25}{15}$

Note: In Problems 4 and 5 the teacher should insist on the form of solution demonstrated in the text.

4. a. $n = 8$

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b. $n = 24$

c. $n = 30$

5. a. $s = 15$

b. $s = 42$

c. $s = 56$

6. It will be $12\frac{1}{2}$ inches long. The ratio of the width to the length will be unchanged. The proportion is $\frac{4}{5} = \frac{10}{l}$.

7. The rate is the ratio of money to time. The proportion is $\frac{135}{40} = \frac{p}{60}$. $p = 202.50$.

8. a. 3 cups butter

2 cups sugar

6 eggs

$4\frac{1}{2}$ cups flour

3 tsp. vanilla

b. The ratio is $\frac{3}{2}$.

$1\frac{1}{2}$ cups butter

1 cup sugar

3 eggs

$2\frac{1}{4}$ cups flour

$1\frac{1}{2}$ tsp. vanilla

9. The ratio of the number of doughnuts to the number of cents they cost will be unchanged. The proportion is $\frac{12}{50} = \frac{10}{c}$. $c = 42$.

10. The ratio compares the number of candy bars with the number of cents they cost. $\frac{3}{25} = \frac{12}{x}$. $x = 100$, cost $1.

11. The ratio is the number of dollars to the number of bricks. $\frac{14}{1000} = \frac{c}{2500}$. $c = 35$, cost $35.

12. b. $v = 18$

c. $w = 35$

d. $v = 85\frac{5}{7}$

e. $w = 116\frac{2}{3}$

13. a , c , d , e , are equal

14. We compare the number of pints of pigment with the number of gallons of paint.

$\frac{2}{3} = \frac{x}{2}$ $3x = 4$, $x = 1\frac{1}{3}$

Mr. Jones will need $1\frac{1}{3}$ pints of pigment.

*15. a. $\frac{2}{EF} = \frac{4}{6}$ b. $\frac{DE}{5} = \frac{6}{4}$

EF = 3 inches DE = $7\frac{1}{2}$ inches

(No extended treatment of similar triangles seems to be appropriate at this time.)

10-3. Percent--A Special Kind of Ratio.

The meaning of percent is based on the idea that "c" % means $\frac{c}{100} = c \times \frac{1}{100}$. All three "kinds" of percent problems are introduced informally with numbers that are easily handled. You will notice that the three "cases" of percent are not referred to in this textbook. Instead, all problems are set up in the form of the proportion $\frac{a}{b} = \frac{x}{100}$. The method of solutions of proportions should be that of Section 10-2 which uses the **Comparison Property** and definition of rational numbers.

Exercises 10-3a emphasize writing percents first as fractions with 100 as the denominator, and then with the % symbol. Another point of emphasis is that 100 % stands for the number one or the whole of a quantity.

Answers to Exercises 10-3a

(Class Discussion)

1. $\frac{7}{100}$, $\frac{13}{100}$, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{37}{100}$, $\frac{1}{2}$, $\frac{3}{4}$, 1 , $\frac{3}{2}$

2. 0.17 , 0.25 , 0.65 , 0.1 , 0.08 , 1 , 1.5 , 2

Answers to Exercises 10-3b

1. a. $\frac{1}{10}$

b. 10 %

c. $\frac{1}{5}$, $\frac{7}{20}$, $\frac{3}{10}$, $\frac{1}{20}$

d. 1

e. 20 %

f. 35 %

g. 30 %

h. 5 %

i. 100 %

2. a. --

b. 40 %

c. $160 \times \frac{1}{100}$, 160 %

d. $\frac{70}{100}$, $70 \times \frac{1}{100}$, 70 %

e. $\frac{75}{100}$, $75 \times \frac{1}{100}$, 75 %

f. $\frac{240}{100}$, $240 \times \frac{1}{100}$, 240 %

3. a. --

b. 400 %

c. 78×0.01 , 78 %

d. 0.90 , 90×0.01 , 90 %

e. 8.00 , 800×0.01 , 800 %

f. 1.20 , 120×0.01 , 120 %

4. a.	Pencil	$\frac{12}{50}$	or	$\frac{24}{100}$	24 %
	Ice Cream	$\frac{10}{50}$	or	$\frac{20}{100}$	20 %
	S. Sch. Coll.	$\frac{15}{50}$	or	$\frac{30}{100}$	30 %
	Savings	$\frac{13}{50}$	or	$\frac{26}{100}$	26 %

b. $\frac{50}{50} = 1$

c. 100 %

d. The ratios total 1 , and the percents total 100 %.

5. a. $\frac{6}{10}$ or 60 %

b. 40 %

c. 100 %

6. a. All wool

b. Everyone has attended

c. Everyone in favor of a picnic

The second lesson in 10-3 emphasizes the use of percent for purposes of comparison, and for giving information in more usable form. More is done with this in Section 10-5. Exercises 10-3a and 10-3b should be done in one day.

Answers to Exercises 10-3c

1. a. 60 %
   b. 40 %
   c. 100 %

2. a. 50 %
   b. Decrease

3. a. 65 %
   b. Higher than the other two.

The third lesson in 10-3 is based on the idea that the study of percent is valuable if the pupil can use the concept to solve problems from everyday experience, and can understand and interpret data expressed in percent. Pupils are given all three cases of percent in the camp story. The solutions all follow the pattern $\frac{a}{b} = \frac{c}{100}$. In class discussion the estimation of a reasonable answer helps the pupil to understand the relationships among a, b and c in various situations, and also serves as a check on an answer.

The Comparison Property from Chapter 8 is used frequently in the solution of problems in this section. The Comparison Property is:

If $\frac{a}{b} = \frac{c}{d}$ and $b \neq 0$, $d \neq 0$, then $a \cdot d = b \cdot c$.

In Exercises 10-3d the importance of estimating and asking if an answer is sensible and reasonable should be emphasized. Question 1(e) picks an incorrect answer simply to focus attention upon how absurd some answers may be. Pupils should avoid falling into the habit of being satisfied with any result they produce. Each answer must be closely examined in the light of the problem.

Answers to Exercises 10-3d
(Class Discussion)

1. a. $\frac{8}{100}$ is the fraction for 8 %

   b. $\frac{n}{250}$ is the ratio of the number of pupils, n, who fail to use their passes to the total number of pupils who buy passes.

   c. The ratios in (a) and (b) are the same because the problem states that 8 % of the pupils were pupils who failed to use their passes.

   d. 20 pupils

   e. No. $\frac{8}{100}$ or 8 % is less than 10 % and 10 % of 250 is 25. This is much less than 200. (Note: Accept any logical response.)

2. a. $\frac{120}{300} = \frac{2}{5}$

   b. x %

   c. $\frac{x}{100}$

   d. $\frac{2}{5} = \frac{x}{100}$ or $\frac{120}{300} = \frac{x}{100}$

   e. $x = 40$

   f. 40 % Yes (Accept any logical reason.)

Answers to Exercises 10-3e

1. a. 30 pupils

   b. $\frac{30}{600} = \frac{x}{100}$, 5 %

   c. 6 pupils

   d. 60 pupils

2. $\frac{36}{600} = \frac{x}{100}$, 6 %

3. a. $\frac{150}{600} = \frac{x}{100}$, 25 %

   b. 75 %

4. a. $\frac{3}{30} = \frac{1}{10}$

   b. $\frac{1}{10} = \frac{x}{100}$, 10%

5. $\frac{4}{100} = \frac{22}{x}$ $x = 550$

6. a. $\frac{50}{100} = \frac{n}{75}$ $n = 37.5$

   b. $\frac{n}{100} = \frac{12}{48}$ $n = 25$

   c. $\frac{75}{100} = \frac{150}{n}$ $n = 200$

   d. $\frac{125}{100} = \frac{100}{n}$ $n = 80$

   e. $\frac{29}{100} = \frac{n}{240}$ $n = 69.6$

   f. $\frac{n}{100} = \frac{2}{40}$ $n = 5$

   g. $\frac{n}{100} = \frac{30}{25}$ $n = 120$

7. $72.

10-4. Ratio as a Percent, a Decimal, a Fraction.

In this section, the work on fractional and decimal equivalents for percent emphasizes that the equivalents are different names for the same number. The bar over the 5 , as in $0.5\overline{5}$, indicates that this is a repeating decimal. The sign $\approx$ for "approximately equal" should be used whenever the limitations of notation require it. Students should not write that $\frac{1}{6}$ equals 16.7 % since the two ratios are not the same. Rather $\frac{1}{6} \approx 16.7\ \%$. We wish here to emphasize these equivalent forms, such as:

$$32\% = \frac{32}{100} = 0.32$$

(Percent) (Fraction) (Decimal)

Answers to Exercises 10-4a
(Class Discussion)

1. 0.17 , 0.02 , 0.035 , 0.65 , 1.15
2. $\frac{1}{4}$, $\frac{1}{10}$, $\frac{1}{20}$, $\frac{13}{20}$, $\frac{11}{10}$, 2
3. 45 % , 4.5 % , 450 % , 42.5 % , 425 %

Answers to Exercises 10-4b

1. a. $\frac{75}{100} = 75\ \%$

   b. $\frac{80}{100} = 80\ \%$

   c. $\frac{70}{100} = 70\ \%$

   d. $\frac{76}{100} = 76\ \%$

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e. $\frac{55}{100} = 55\%$

f. $\frac{36}{100} = 36\%$

g. $\frac{85}{100} = 85\%$

h. $\frac{60}{100} = 60\%$

2. a. .33
   b. .83
   c. .38
   d. .78
   e. .55
   f. .88
   g. .68
   h. .45

3. a. 33%
   b. 83%
   c. 38%
   d. 78%
   e. 55%
   f. 88%
   g. 68%
   h. 45%

4. a. .375
   b. .667
   c. .889
   d. .967
   e. .833
   f. .892
   g. .257
   h. .838

5. a. 37.5%
   b. 66.7%
   c. 88.9%
   d. 96.7%
   e. 83.3%
   f. 89.2%
   g. 25.7%
   h. 83.8%

You must round the decimal to thousandths.

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Answers to Exercises 10-4c

1.	Fraction in Simplest Form	Hundred as Denominator	Decimal	Percent
a.				
b.		$\frac{25}{100}$	.25	25 %
c.	$\frac{3}{4}$		.75	75 %
d.	$\frac{1}{5}$	$\frac{20}{100}$		20 %
e.	$\frac{2}{5}$	$\frac{40}{100}$	.40	
f.	$\frac{3}{5}$		.60	60 %
g.		$\frac{80}{100}$	.80	80 %
h.	$\frac{1}{3}$	$\frac{33\frac{1}{3}}{100}$		$33\frac{1}{3}$ % or 33.3 %
i.	$\frac{7}{10}$		.70	70 %
j.	$\frac{2}{3}$	$\frac{66\frac{2}{3}}{100}$		$66\frac{2}{3}$ % or 66.7 %
k.		$\frac{30}{100}$	.30	30 %
l.	$\frac{1}{10}$	$\frac{10}{100}$		10 %
m.	$\frac{9}{10}$	$\frac{90}{100}$	.90	
n.		$\frac{12\frac{1}{2}}{100}$	.125	12.5 % or $12\frac{1}{2}$ %
o.	$\frac{3}{1}$ or 3		3 or 3.00	300 %
p.	$\frac{3}{8}$	$\frac{37\frac{1}{2}}{100}$		37.5 % or $37\frac{1}{2}$ %

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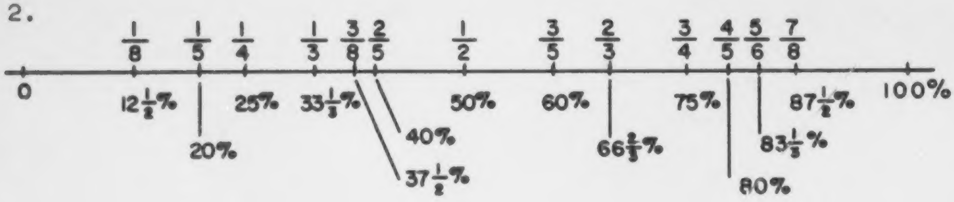
	Fraction in Simplest Form	Hundred as Denominator	Decimal	Percent
q.	$\frac{3}{2}$	$\frac{150}{100}$	1.5	
r.	$\frac{5}{8}$		.625	62.5 % or $62\frac{1}{2}$ %
s.	$\frac{1}{100}$	$\frac{1}{100}$		1 %
t.		$\frac{87\frac{1}{2}}{100}$	.875	87.5 % or $87\frac{1}{2}$ %
u.	1	$\frac{100}{100}$	1 or 1.00	
v.	$\frac{1}{6}$		$.1\overline{6}$	$16\frac{2}{3}$ % or 16.7 %
w.		$\frac{83\frac{1}{3}}{100}$	$.83\overline{3}$	83.3 % or $83\frac{1}{3}$ %

2.


$\frac{1}{8}$
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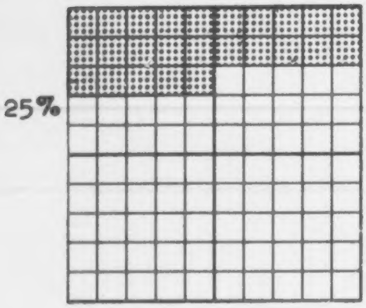
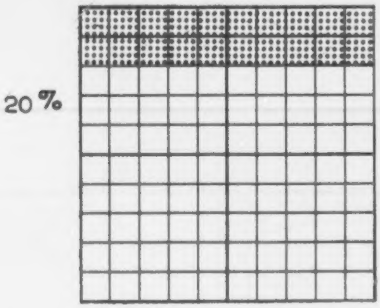
$\frac{5}{6}$
$\frac{7}{8}$
0
$12\frac{1}{2}$%
20%
25%
$33\frac{1}{3}$%
40%
$37\frac{1}{2}$%
50%
60%
$66\frac{2}{3}$%
75%
80%
$83\frac{1}{3}$%
$87\frac{1}{2}$%
100%

3.





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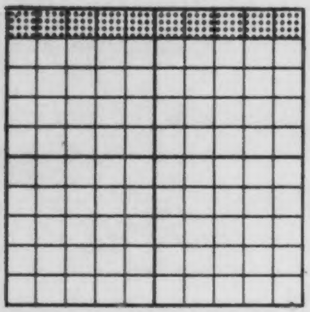
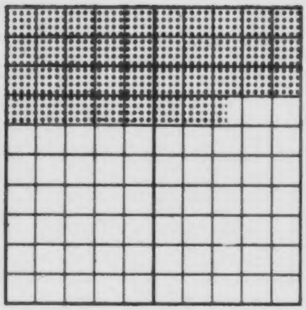
10%



$37\frac{1}{2}$%



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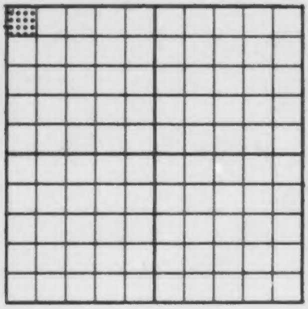

4. a. $\frac{8}{25}$ c. $1\frac{1}{5}$ or $\frac{6}{5}$

   b. $\frac{9}{10}$

5. a. 52 %   c. 95 %

   b. 35 %   d. 30 %

10-5. Applications of Percent.

Specific applications of percent are introduced so that pupils may use their new skills with percent and proportion. Problems which arise in social situations dealing with budgets, commission, discount, sports records, increase, decrease, and simple interest are sampled. Students should realize that every percent problem falls into the same pattern. Two ratios are equal, as in $\frac{a}{b} = \frac{c}{100}$.

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The examples in the text are designed for class discussion. In some cases it may be helpful to devise additional problems for more class discussion before the students are ready to proceed on their own.

There are certain terms used in business which the pupils will need to understand. These should be discussed in class. Diagrams may be used to help clarify relationships.

The sets of exercises are long enough to provide examples for class discussion and for written assignments.

Fractional percents are developed and students are given experience in expressing ratios to the nearest tenth of a percent. If this is too difficult for some classes, the teacher may wish to omit the problems in which it arises.

Many students and teachers would prefer to do the example 32% of $410 using decimals. It is possible to think of 32% of 410 as 0.32×410 since in "a fraction of a number" the "of" means times. However, the ratio method is useful in every type of percent problem, and we believe that the student will be less confused if the teacher uses the same approach for all problems.

Answers to Exercises 10-5a

1. The total income is distributed, so the sum of the percents must be 100% in each case.

2. 22%

3. a. $\frac{22}{100}$
   b. $\frac{f}{450}$
   c. Estimate about $90
   d. $\frac{f}{450} = \frac{22}{100}$ $f = 99$
   e. The actual answer is a little more than the estimate which we knew was low.

4. a. 16%
   b. $\frac{s}{550} = \frac{16}{100}$ $s = 88$
   c. $\frac{1}{10}$ of 550 is 55

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$\frac{1}{5}$ of 550 is 110

The answer to b. is between the two.

5. $\frac{13}{100} = \frac{p}{300}$ $p = 39$

6. $\frac{7}{100} = \frac{t}{350}$ $t = 24.50$

7. $\frac{28}{100} = \frac{h}{450}$ $h = 126$

8. $\frac{5}{100} = \frac{s}{300}$ $s = 15$

9. $\frac{150}{575} = \frac{x}{100}$ $x \approx 26\%$ This is less than 28% , the allowance in the table for an income of $550.

*10. $\frac{95}{m} = \frac{25}{100}$ $m = 380$

Answers to Exercises 10-5b

1. a. $\frac{n}{40} = \frac{100}{100}$ $n = 40$ c. $\frac{n}{40} = \frac{50}{100}$ $n = 20$
   b. $\frac{n}{40} = \frac{80}{100}$ $n = 32$ d. $\frac{n}{40} = \frac{65}{100}$ $n = 26$

2. a. $\frac{30}{40} = \frac{p}{100}$ $p = 75\%$ c. $\frac{18}{40} = \frac{p}{100}$ $p = 45\%$
   b. $\frac{36}{40} = \frac{p}{100}$ $p = 90\%$ d. $\frac{38}{40} = \frac{p}{100}$ $p = 95\%$

3. a. $\frac{4}{100} = \frac{t}{17.50}$ $t = .70$
   b. $\frac{4}{100} = \frac{t}{49.50}$ $t = 1.98$

4. $\frac{c}{27,500} = \frac{5}{100}$ $c = 1,375$

5. $\frac{9000}{s} = \frac{5}{100}$ $s = 180,000$

6. $\frac{25.50}{85.00} = \frac{c}{100}$ $c = 30\%$

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7. $\frac{1}{100} = \frac{c}{658,000 + 482,000}$ $c = 11,400$ Yes, adequate.

8. a. $\frac{27}{100} = \frac{d}{5.98}$

Discount $d = 1.61$

$s = \$5.98 - \$1.61 = \$4.37$

b. $\frac{27}{100} = \frac{d}{3.75}$

$d = 1.01$

$s = \$2.74$

9. a. 1117 total enrollment

b. $\frac{380}{1117} = \frac{p}{100}$ $p \approx 34.0\,\%$

c. $\frac{385}{1117} = \frac{p}{100}$ $p \approx 34.5\,\%$

d. $\frac{352}{1117} = \frac{p}{100}$ $p \approx 31.5\,\%$

e. 100%

10. $\frac{96}{t} = \frac{4}{100}$ $t = 2400$

Answers to Exercises 10-5c

1. George $\frac{19}{65} \approx .292 \approx 29.2\,\%$

Max $\frac{22}{70} \approx .314 \approx 31.4\,\%$

Bill $\frac{21}{73} \approx .288 \approx 28.8\,\%$

Tom $\frac{21}{60} = .350 = 35.0\,\%$ best average

2. $\frac{p}{80} = \frac{4.6}{100}$ $p = 3.7$

3. $\frac{3330}{c} = \frac{18.5}{100}$ $c = 18,000$

4. $\frac{p}{100} = \frac{450}{5200}$ $p = 8.7\,\%$

Answers to Exercises 10-5d

1. a. $\frac{p}{100} = \frac{20}{176}$ $p \approx 11.4\%$

   b. $\frac{p}{100} = \frac{37}{176}$ $p \approx 21.0\%$

   c. $\frac{p}{100} = \frac{65}{176}$ $p \approx 36.9\%$

   d. $\frac{p}{100} = \frac{40}{176}$ $p \approx 22.7\%$

   e. $\frac{p}{100} = \frac{14}{176}$ $p \approx 8.0\%$ (.0795)

   f. 100 % Yes

2. $\frac{13}{65} = \frac{p}{100}$ $p = 20\%$ increase

3. $\frac{16}{160} = \frac{p}{100}$ $p = 10\%$ decrease

4. $\frac{I}{1240} = \frac{25}{100}$ $I = 310$ enrollment
   $1240 + 310 = 1550$

5. $\frac{4.50}{14} = \frac{p}{100}$ $p = 32.1\%$ decrease

6. a. $\frac{p}{100} = \frac{4850}{970,000}$ $p = .5\%$ or $\frac{1}{2}\%$

   b. $\frac{.5}{100} = \frac{c}{847,500}$ $c = 4,237.50$

7. $\frac{p}{100} = \frac{5}{60}$ $p = 8.3\%$ or $8\frac{1}{3}\%$

8. Answers will vary.

9. a. First player: $\frac{25}{83} = .301$ or 30.1%

   Second player: $\frac{42}{143} = .294$ or 29.4%

   b. The first player.

10. $\frac{p}{100} = \frac{222}{790}$ $p \approx 28.1\%$ increase

11. $\frac{p}{100} = \frac{150}{1050}$ $p \approx 14.3\%$ increase

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12. $\frac{p}{100} = \frac{25}{450}$ $p \approx 5.6\ \%$ decrease

13. a. 15 lbs. 2 oz.

b. Since 7 lbs. 9 oz. = 121 oz.
increase 17 lbs. - 7 lbs. 9 oz. = 151 oz.
$\frac{p}{100} = \frac{151}{121}$ $p \approx 125\ \%$ increase

14. $\frac{p}{100} = \frac{460}{1490}$ $p \approx 31\ \%$

15. (b) is correct. The 1960 wages are less than the 1958 wages. Students may understand this result better if they see what happens to a particular amount like $100. If $100 is decreased by 20 %, it is $80. $80 increased by 20 % is only $96.

Answers to Exercises 10-5e

1. $\frac{I}{1800} = \frac{4}{100}$ $I = 72$

2. $\frac{r}{100} = \frac{75}{1250}$ $r = 6\ \%$

3. $\frac{I}{900} = \frac{3.5}{100}$ $I = 31.50$

4. $\frac{42.5}{p} = \frac{5}{100}$ $p = 850$

5. $\frac{I}{250} = \frac{4}{100}$ $I = 10$

6. $105

7. $3\frac{1}{2}\ \%$

8. $40

9. $350 + $14 = $364

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*10. \$364 + \$14.56 = \$378.56

10-7. Chapter Review.

Answers to Exercises 10-7

1. a. $\frac{22}{24} = \frac{11}{12}$

   b. $\frac{63}{56} = \frac{9}{8}$

2. a. Equal   c. Equal
   b. Not equal   d. Equal

3. a. $c = 30$ c. $c = 15$
   b. $c = 57$ d. $c = \frac{400}{27}$ or $14\frac{22}{27}$

4. $2\frac{1}{4}$ inches high $\frac{4}{4.5} = \frac{2}{h}$

5. $16\frac{1}{2}$ inches, 12 inches

6. a. 2 %   c. $66\frac{2}{3}$ %
   b. 5.5 %   d. $12\frac{1}{2}$ % or 12.5%

7. 20%

8. \$2700

9. About 1415 students

10. \$980

11. \$8.40

12. 100 % increase

13. If $\frac{a}{b} = \frac{c}{d}$ $(b , d \neq 0)$, then $a \cdot d = b \cdot c$

If $\frac{15}{27} = \frac{20}{36}$, then $15 \cdot 36 = 27 \cdot 20$

$540 = 540$

14. Rm. 106 .813 or 81.3 %

Rm. 107 .833 or 83.3 %

Rm. 107 had the better score.

15. $31.50

10-8. Cumulative Review.

Answers to Exercises 10-8

1. a. True
   b. True
   c. True

2. Base Four

3. a. 32_{five}
   b. 21_{eight}
   c. 10001_{two}

4. $31 \cdot 5 \cdot 2 = 31 \cdot (5 \cdot 2)$
   $= 31 \cdot 10$
   $= 310$

5. a. Eight is less than twelve.
   b. Thirty-four is greater than thirty-two.
   c. Five is greater than three and three is greater than two.

6. $a + b = b + a$

7. a. Infinitely many
   b. One
   c. One

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8. a. The interior of $\angle AEC$
   b. $\overleftrightarrow{BC}$

9. a. $\frac{4}{10}$
   b. 13
   c. $\frac{9}{11}$

10. a. 8.204
    b. 2.287

11. a. 80 %
    b. 2.6 %
    c. $83\frac{1}{3}$ %
    d. $37\frac{1}{2}$ %

12. No. 72% > 67.6

13. a. 4
    b. 11
    c. 1
    d. 0
    e. $\frac{13}{7}$
    f. $\frac{1}{6}$
    g. 12
    h. $\frac{1}{6}$
    i. 60
    j. 100

14. a. .0075
    b. .0014
    c. .42
    d. .1304

15. a. .104
    b. .47
    c. 151
    d. 1.53

Sample Test Questions for Chapter 10

Note: Teachers should construct their own tests, using carefully selected items from those given here and from items of their own. Careful attention should be given to difficulty of items and time required to complete the tests.

True-False

(T) 1. $\frac{8}{25}$ is another name for the number 32%.

(T) 2. An increase in the price of an item from \$20 to \$28 is an increase of 40%.

(F) 3. If a class has a total of 32 pupils, 20 of them boys, the number of boys is 60% of the number of pupils in the class.

(T) 4. Five percent of \$150 is the same amount of money as 7.5% of \$100.

(F) 5. 62.5% and $\frac{5}{8}$ are names for two different numbers.

Multiple Choice

1. Six percent of \$350 is

    a. \$210.00
    b. \$21.00
    c. \$2.10
    d. \$2100.
    e. None of these

    b.

2. If 8% of the number 5400 is computed, the correct answer is

    a. More than 30 but less than 90
    b. More than 3 but less than 5
    c. More than 54 but less than 540
    d. More than 540 but less than 1000
    e. None of these

    c.

3. If $\frac{1}{2}$% of \$320.00 is computed, the answer is

    a. \$16
    b. \$160
    c. \$3.20
    d. \$1.60
    e. None of these

    d.

4. In a class of 42 pupils there are 25 boys. The number of boys is what percent (nearest whole percent) of the number of pupils?

    a. 60%
    b. 59%
    c. 58%
    d. 61%
    e. None of these

    a.

5. In the class of 42 pupils there are 17 girls. The number of girls is what percent (rounded to the nearest tenth of a percent) of the number of pupils?

a. 40.4%
b. 40.6%
c. 39.9%
d. 40.7%
e. None of these

e.

6. The interest on \$650 for one year at 4% is

a. \$2600
b. \$1625
c. \$26
d. \$16.25
e. None of these

c.

7. Another numeral for .05 is:

a. $\frac{1}{20}$%
b. $\frac{1}{5}$%
c. $\frac{1}{2}$%
d. 5%
e. 50%

d.

8. Which of the following is another name for $\frac{24}{25}$?

a. 0.0096%
b. 0.096%
c. 24%
d. 96%
e. 99%

d.

9. A team won 3 games out of 5 played. Which of the following proportions should be used to find the percent of games won?

a. $\frac{100}{n} = \frac{3}{5}$
b. $\frac{n}{5} = \frac{3}{100}$
c. $\frac{n}{3} = \frac{5}{100}$
d. $\frac{n}{100} = \frac{5}{3}$
e. $\frac{3}{5} = \frac{n}{100}$

e.

10. A budget used by the Smith family recommended that 27% of the take-home pay be set aside for food. The take-home pay for the Smiths was \$350 per month. How much should be set aside for food each month?

a. \$94.50
b. \$255.50

c. $9.45

d. $25.55

e. Cannot be determined from the information given.

a.

11. Five members of a small group vote against a certain issue; one votes in favor of it. The ratio of the number of votes against the issue to the number in favor of it is:

a. $\frac{1}{5}$

b. 5

c. $\frac{5}{6}$

d. $\frac{6}{1}$

e. $\frac{6}{5}$

b.

12. A class consists of 20 girls and 16 boys. What part of the class is composed of boys?

a. $\frac{4}{5}$

b. $\frac{5}{9}$

c. $\frac{4}{9}$

d. $\frac{9}{4}$

e. $\frac{5}{4}$

c.

Additional Problems

1. What commission does a real estate agent receive for selling a house for $15,400 if his rate of commission is 5 %?

$770

2. The sale price on a dress was $22.80 and the marked price showing on the price tag was $30.00. What was the rate of discount?

24%

3. An increase in rent of 5 % of the present rent will add $3.50 to the monthly rent that Mr. Johnson will pay. What is the monthly rent that Mr. Johnson now pays?

$70

4. A family budget allows 30% of the family income for food. If the monthly income of the family is $423, what amount of money is allowed for food for the month?

$126.90

5. Dorothy was 5 feet tall (to the nearest inch) when school opened in September. In June her height was

5 feet 3 inches. What is the percent of increase in her height?

5%

6. At a certain time of the day a man 6 feet tall casts a shadow 8 feet long. At the same time a tree casts a shadow of 40 feet. How tall is the tree?

30 feet

7. What number n will make the following statement true: $\frac{n}{10} = \frac{12}{15}$?

8

8. What percent is $\frac{7}{12}$? (Express your answer to the nearest tenth of a percent.)

58.3%

9. Joyce weighs 90 pounds and Barbara weighs 80 pounds. What is the ratio of Joyce's weight to Barbara's weight?

$\frac{9}{8}$

10. A sofa marked $200 is sold at a 30 percent discount. What is the sale price?

$140

11. Find the interest on $850 for one year at 6%.

$51

12. Mr. Jones earns as commission 15% of his sales. His commission in May was $375. What was the amount of his sales in May?

$2500

The preliminary edition of The Introduction to Secondary School Mathematics was prepared at the writing session held at Stanford University during the summer of 1960 and was based upon Volume I of Mathematics for Junior High School. Parts 1 and 2 of The Introduction to Secondary School Mathematics were revised at Yale University in the summer of 1961 in accordance with reports of classroom effectiveness. Part 4, based on selected chapters of Volume II of Mathematics for Junior High School was prepared at the same time. The present revision, taking into account the classroom experience with the earlier editions during the academic years of 1961-62, was prepared at Stanford University during the summer of 1962.

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